Bounding the dependence measures for spatial uncertainties

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Outline

Outline

Introduction

Representation of spatial uncertainty in numerical modelling

→ **Needed at *input* side of an analysis:**

E.g., material properties, loads

Framework to capture the available information?

- (Spatial variation of) Bounds on parameter
- Dependency between local values of parameter: limit the possible realisations in between the bounds.

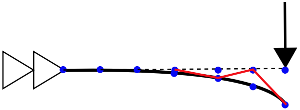
→ **Needed at *output* side of an analysis:**

E.g., displacement field, temperature field

How to accurately represent deterministic realisation?

How to represent uncertainty without conservatism?

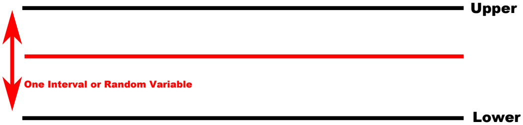
- Predicting correct bounds on local values
- Avoid impossible realisations



Introduction

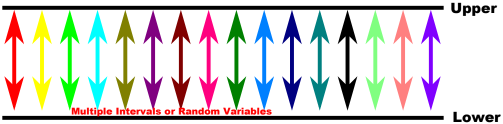
Representation of spatial uncertainty in numerical modelling

- Assuming uniformity or homogeneity, total dependence:



This representation is not sufficient.

- Assuming total independence:



This representation is not necessary and infeasible.

Introduction

Representation of spatial uncertainty in numerical modelling

What is needed?

- **Dependence measure**
- **Realisations**
- **Uncertainty propagation**
  - Types:
    - scalar $\rightarrow$ field
    - field $\rightarrow$ scalar
    - field $\rightarrow$ field
  - Optimisation problem: what to optimise?
- **Representation of output**
  - Bounds with dependence measure $\rightarrow$ realisations
  - Analytical formulation

Outline

1. Introduction
2. Dependence measures
   - Probabilistic dependence measures
   - Non-probabilistic dependence measures
3. Spatial dependence
4. Academic example

Dependence measures

"We must be clearer about the abused word *dependence* and its relatives." [Drouet Mari and Kotz, 2001]

- 'Perfect' dependence: $x = a \to y = b$
- 'Flexible' dependence: $x \in x^s \subset x^S \to y \in y^s \subset y^S$

"... an essential part of uncertainty analysis is the analysis of dependence." [Kurowicka and Cooke, 2006]

"..., the only means to establish a relationship between variables was to deduce a causative connection. There was no way to discuss - let alone measure - the association between variables that lack a cause-effect relationship." [Rodgers and Nicewander, 1988]

Dependence measures

Probabilistic dependence measures

Probabilistic dependence measures:

- Scalar aggregate (global) measure:
  - Product moment correlation
  - Rank correlation
  - Kendall’s tau
  - Relative entropy
- Thorough (local) measure:
  - Copula

Dependence measures

Probabilistic dependence measures

Product moment correlation

$$\rho(X,Y) \equiv \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$$

- Depends on marginal distributions
- $-1 \leq \rho(X,Y) \leq 1$
- min & max (not necessarily -1 and 1) when countermonotonic or comonotonic
- Invariant under linear strictly increasing transformations
- X and Y are independent $\rightarrow \rho(X,Y) = 0$

Dependence measures

Rank correlation

$$\rho_r(X,Y) \equiv \rho(F_X(X), F_Y(Y))$$

- Independent of marginal distributions
- $-1 \leq \rho_r(X,Y) \leq 1$
- min & max when countermonotonic or comonotonic
- Invariant under non-linear strictly increasing transformations
- X and Y are independent $\rightarrow \rho_r(X,Y) = 0$

Elucidation

$$\pi_{\frac{1}{2}}(X,Y) \equiv P\left(F_X(X) > \frac{1}{2} | F_Y(Y) > \frac{1}{2}\right)$$

- $\pi_{\frac{1}{2}}(X,Y) = 0 \rightarrow \rho_r(X,Y) = -1$
- $\pi_{\frac{1}{2}}(X,Y) = 1/2 \rightarrow \rho_r(X,Y) = 0$
- $\pi_{\frac{1}{2}}(X,Y) = 1 \rightarrow \rho_r(X,Y) = 1$

Dependence measures

Probabilistic dependence measures

Kendall's tau

$$\tau(X,Y) \equiv P\left[(X_1 - X_2)(Y_1 - Y_2) \geq 0\right] - P\left[(X_1 - X_2)(Y_1 - Y_2) < 0\right]$$

- Independent of marginal distributions
- $-1 \leq \tau(X,Y) \leq 1$
- Invariant under non-linear strictly increasing transformations
- X and Y are independent $\rightarrow \tau(X,Y) = 0$

Dependence measures

Probabilistic dependence measures

Relative entropy

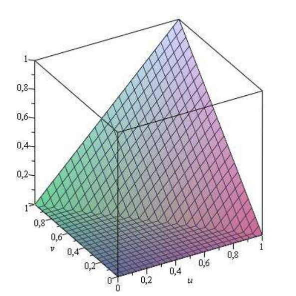
$$\delta_{X,Y} \equiv \int\int f(x,y)\ log\left(\frac{f(x,y)}{f_1(x)f_2(y)}\right) dxdy$$

- Entropy of $f(x,y)$ is compared with maximum attainable entropy when X and Y are independent
- X and Y are independent $\rightarrow \delta_{X,Y} = 0$
- X and Y are perfect dependent $\rightarrow \delta_{X,Y}$ tends to ∞

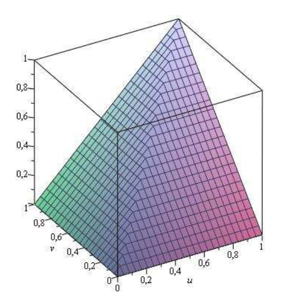
Copula

$$F_{XY}(x, y) = C(F_X(x), F_Y(y))$$

- Seperates effect of dependence and marginal distributions
- Cumulative distribution function
  with uniform marginals on $[0, 1]$
- $C_L = max(u + v - 1, 0) \leq C(u, v) \leq C_U = min(u, v)$
  with $(u, v) \in [0, 1]^2$



C_L



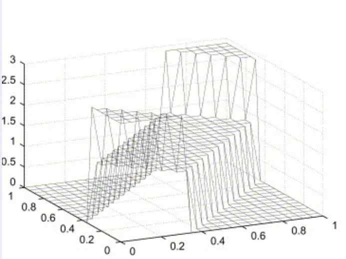
C_U

Dependence measures

Probabilistic dependence measures

Most promising two copulae

- Diagonal band copula



- Minimum information copula: unique bivariate joint distribution with
  - marginal distribution uniform on $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$
  - rank correlation $\rho_r \in [-1, 1]$
  - minimal information relative to uniform distribution among all distributions with rank correlation ρ_r
  - no closed functional form

Dependence measures

Non-probabilistic dependence measures

Non-probabilistic dependence measures:

- Independence and noninteraction dominate

Hartley measure of uncertainty

$$H(r_E) = \log_2 |E|, \quad r_E(x) = \begin{cases} 1 & \text{when } x \in E \\ 0 & \text{when } x \notin E \end{cases}$$

Information transmission

$$T_H(X,Y) = H(X) + H(Y) - H(X \times Y)$$

- Noninteractive $\rightarrow T_H(X,Y) = 0$
- $H(X|Y) = H(Y|X) = 0 \rightarrow$ maximum of $T_H(X,Y)$

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Outline

Spatial dependence

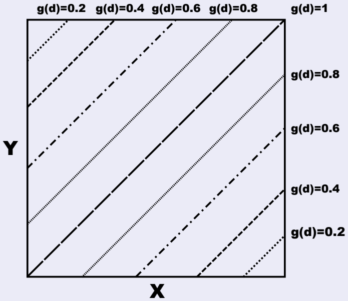
Link to spatial dependence

Link to spatial dependence:

- A dependence description as a function of the distance between points
  - Prob: Covariance structure of Random Field
    - Well developed discretisation methods
    - Criteria to assess the discretisation
  - Non-prob: Domain of influence

Domain of influence

- Inspired by diagonal band copula



- d is distance between two points
- $g(d) \to$ set $R \subseteq X \times Y$ of possible values
- Other shapes for R possible

Domain of influence: $g(d)$

- $g(d) : d \to \lfloor -\infty, 1 \rceil$
- $g(d) \leq 0$ then $R = X \times Y$
- $g(d) = 1$ then R reduces to the single line $X = Y$.
- Conditions on $g(d)$: $\begin{cases} g(0) = 1 & \\ g(d_1) > g(d_2) & \text{for } d_1 < d_2 \end{cases}$
- Example: $g(d) = 1 - \dfrac{d}{a}$

  $a > 0$, a parameter specifying the domain of influence.
  - $d < a$ then $R \subset X \times Y$
  - $d > a$ then $R = X \times Y$

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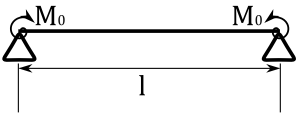
Outline

Academic example

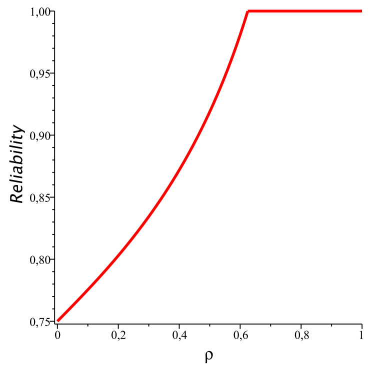
Academic example:



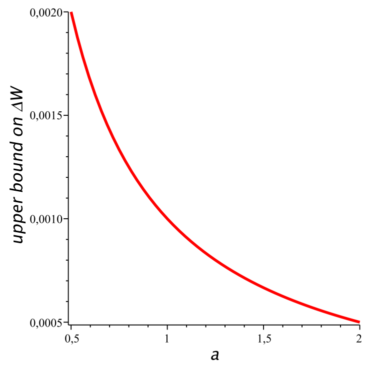
- Input:
  - uncertain flexibility, 2 point discretisation
  - interval on correlation length
  - interval on domain of influence a
- Output: displacement difference $\Delta W(x_1, x_2)$
  - Prob: reliability
  - Non-prob: upper bound

Academic example

Reliability



Upperbound



Conclusions

What was given:

- Overview of dependence measures
- Link between set definition and spatial uncertainty

What is still needed:

- Non-probabilistic dependence measures needed
- Dependence measure $\rightarrow$ realisations:
  - criteria for realisations

http://www.isma-isaac.be/ ISMA - International...

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