Generalized models for uncertainty and imprecision in engineering

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GENERAL SITUATION

Endeavor

- numerical modeling – physical phenomena, structure, and environment

  → prognosis – system behavior, hazards, safety, risk, robustness, economic and social impact, ...

» close to reality
» numerically efficient

Deterministic methods

- deterministic structural parameters
- deterministic computational models

↔ Reality

Uncertainty ?

Imprecision ?

Vagueness ?

Fluctuations ?

Ambiguity ?

Variability ?

Indeterminacy ?

→ Consequences ?

TRADITIONAL MODELING OF UNCERTAINTY

Probability

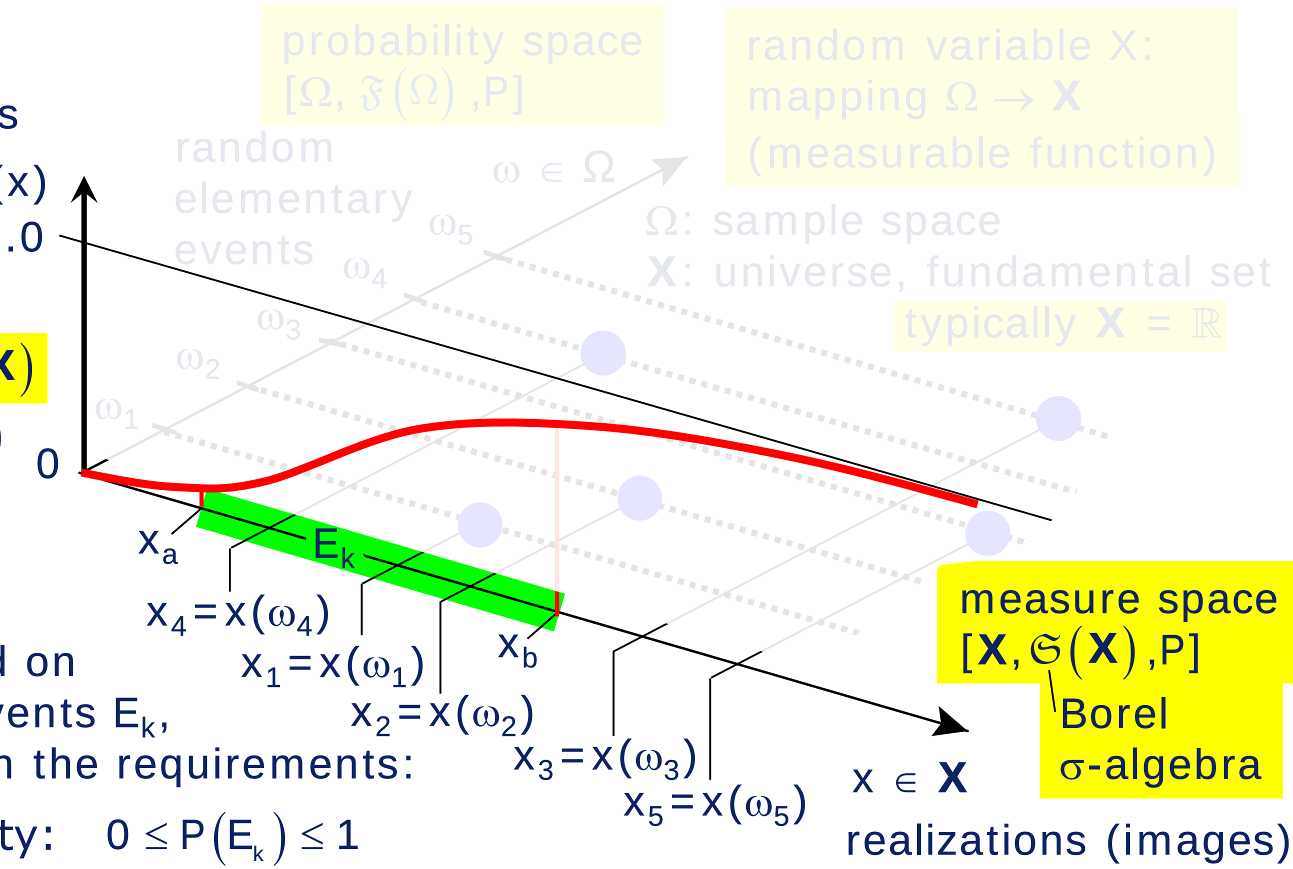
- generation of events

event of interest

E_k: $x_a < X \le x_b$

$\{x \mid x_a < x \le x_b\} \in \mathfrak{S}(\mathbf{X})$

$P(E_k) = F(x_b) - F(x_a)$



- axiomatic definition

P is a real-valued set function defined on a system of sets/events E_k, which complies with the requirements:

» A1 – nonnegetivity: $0 \le P(E_k) \le 1$

» A2 – normalization: $P(\mathbf{X}) = 1$

» A3 – i) additivity: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ for $E_1 \cap E_2 = \varnothing$

ii) σ-additivity: $P\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} P(E_k)$ for $E_k \cap E_j = \varnothing, k \ne j$

SUBJECTIVE PROBABILITIES

Combination of rare data and prior expert knowledge

BAYESian approach

prior distribution

$$f_{\Theta|X_1,\ldots,X_n}\left(\theta \middle| x_1,\ldots,x_n\right) = \frac{\left[\prod_{i=1}^{n} f\left(x_i \middle| \theta\right)\right] \cdot g_\Theta\left(\theta\right)}{\int \left[\prod_{i=1}^{n} f\left(x_i \middle| \theta\right)\right] \cdot g_\Theta\left(\theta\right) d\theta}$$

posterior distribution

» update of expert knowledge by means of data

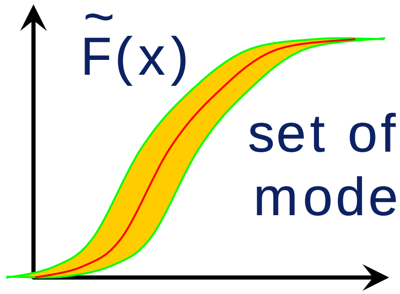
» influence of subjective assessment (prior distribution) decays quickly with increase of sample size

» questions
- evaluation of remaining subjective uncertainty
- consideration of imprecision (data and expert knowledge)
- very small sample size or no data

SOME QUESTIONS

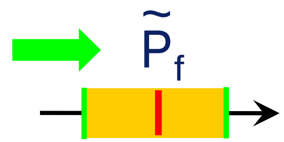
statistical analysis of imprecise and rare data

Is it safe ?



set of plausible models

reliability analysis



$[P_{f,l}, P_{f,r}]$
imprecision
reflected in P_f

Is the reliability analysis still reliable ?

Effects on P_f ?

Sensitivity of P_f to imprecision ?

SOME QUESTIONS

How precise will it be ?

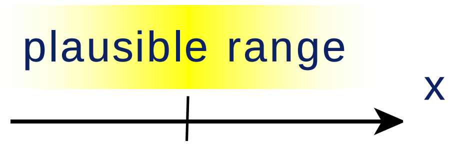
Any concern or doubt ?

modeling, quantification, processing, evaluation, interpretation ?

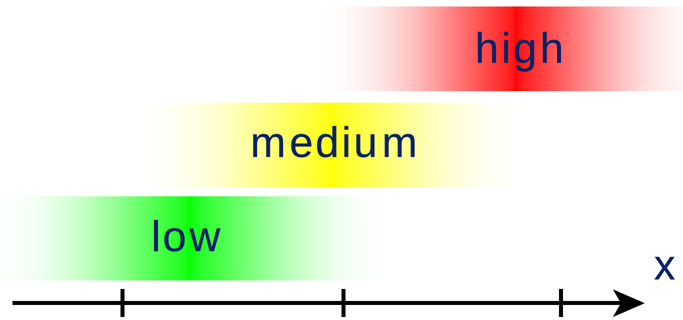
PROBLEMATIC CASES

Summary of examples

- imprecise measurements
- measurement / observation under dubious conditions



- imprecise sample elements
- small samples
- incomplete probabilistic elicitation exercise
- vague / dubious probabilistic information
- changing environmental conditions
- expert assessment / experience
- linguistic assessments



- observations which cannot be separated clearly
- conditional probabilities observed under unclear conditions
- only marginals of a joint distribution available without copula function

➡ mixture of information from different sources and with different characteristics

➡ classification and mathematical modeling ?

CLASSIFICATION AND MODELING

According to sources

- aleatory uncertainty
  - » irreducible uncertainty
  - » property of the system
  - » fluctuations / variability

  stochastic characteristics

  ➡ traditional probabilistic models

- epistemic uncertainty
  - » reducible uncertainty
  - » property of the analyst
  - » lack of knowledge or perception

  collection of all problematic cases, inconsistency of information

  ➡ no specific model

According to information content

- uncertainty
  - » probabilistic information

  ➡ traditional and subjective probabilistic models

- imprecision
  - » non-probabilistic characteristics

  ➡ set-theoretical models

In view of the purpose of the analysis

- averaged results, value ranges, worst case, etc. ?

CLASSIFICATION AND MODELING

Simultaneous appearance of uncertainty and imprecision

- information defies a pure probabilistic modeling
- separate treatment of uncertainty and imprecision in one model
- generalized models combining probabilistics and set theory

➡ **concepts of imprecise probabilities**

» interval probabilities
» sets of probabilities / p-box approach
» random sets
» fuzzy random variables / fuzzy probabilities
» evidence theory / Dempster-Shafer theory

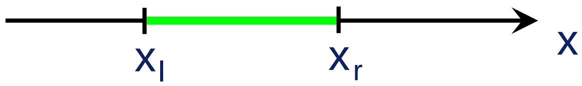
common basic feature:
set of plausible probabilistic models over a range of imprecision
(set of models which agree with the observations)

➡ bounds on probabilities for events of interest

INTERVALS

Mathematical model

- $X = [x_l, x_r] = \{\, x \in \mathbf{X} = \mathbb{R} \mid x_l \le x \le x_r \,\}$ (classical set X)



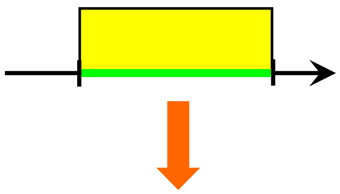
Information content

- possible value range between crisp bounds
- no additional information (on fluctuations etc.)

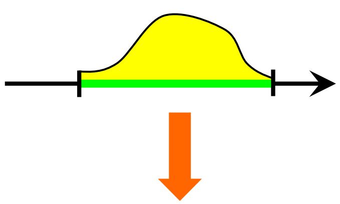
Numerical processing

- option I: enclosure schemes
  - » narrow actual result interval from outside
  - » high numerical efficiency
  - » procedures restricted to specific problems
- option II: global optimization
  - » explicit search for result interval bounds
  - » reasonable or high numerical effort
  - » procedures applicable to a large variety of problems

common suggestion:
assignment of
uniform distributions



engineering analysis

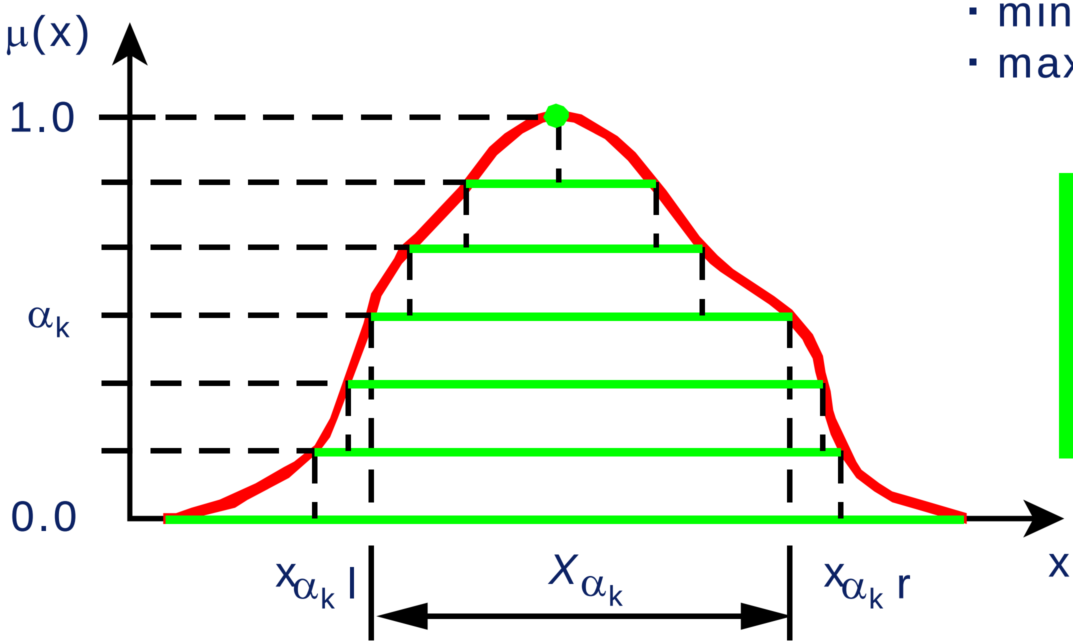


meaning ?
effort ?
does it serve the purpose of the analysis ?

FUZZY SETS

Modeling of imprecision, numerical representation

- α-level set $X_\alpha = \{ x \in \mathbf{X} \mid \mu(x) \geq \alpha \}$
- α-discretization $\tilde{X} = \{ (X_\alpha , \mu (X_\alpha)) \}$



- aggregation of information
  - » max-min operator
    - min operator = t-norm
    - max operator = t-co-norm

$\mu(.)$ not important but analysis with various intensities of imprecision

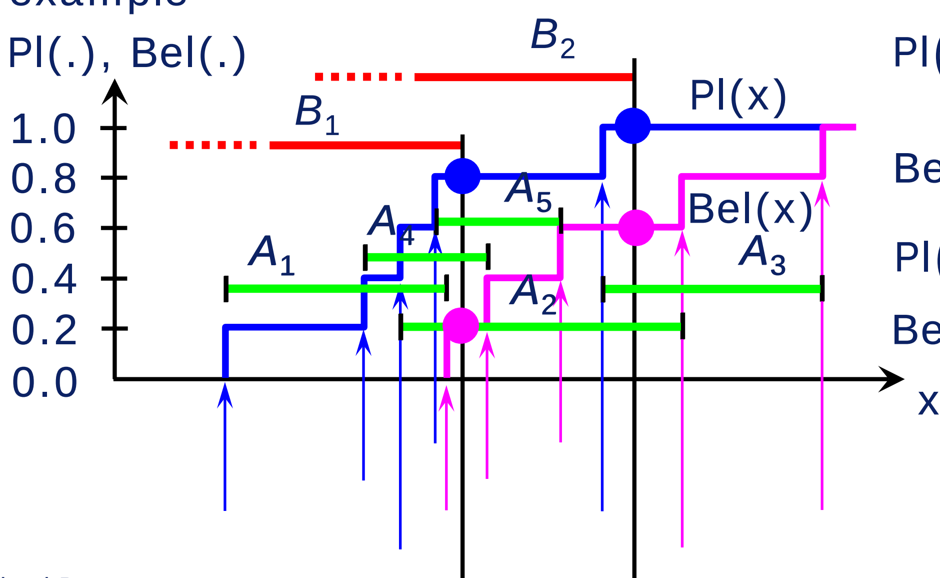
- set of nested intervals of various size

➡ utilization of interval analysis for α-level sets, calculation of result-bounds for each α-level

EVIDENCE THEORY (DEMPSTER–SHAFER)

Mathematical model

- body of evidence with focal subsets A_i (observations), $A_i \in \mathfrak{S}(\mathbf{X}) \; \forall A_i$
- assignment of probability weights to the A_i: basic probability assignment
  $w(A_i)$ with $0 \le w(A_i) \le 1, \; \sum_i w(A_i) = 1$ (A_i sometimes interpreted as random sets)
- plausibility and belief measures for sets $B_j \in \mathfrak{S}(\mathbf{X})$ (events of interest)
  $\mathrm{Pl}(B_j) = \sum_i w(A_i) \big| A_i \cap B_j \neq \emptyset$, $\mathrm{Bel}(B_j) = \sum_i w(A_i) \big| A_i \subseteq B_j$, $\mathrm{Pl}(B_j) + \mathrm{Bel}(B_j^c) = 1$
- example



$\mathrm{Pl}(B_1) = w(A_1) + w(A_2) + w(A_4) + w(A_5)$
$= 0.8$

$\mathrm{Bel}(B_1) = w(A_1) = 0.2$

$\mathrm{Pl}(B_2) = \sum_{i=1}^{5} w(A_i) = 1.0$ $\quad w(A_i) = 0.2, \; i = 1, \ldots, 5$

$\mathrm{Bel}(B_2) = w(A_1) + w(A_4) + w(A_5) = 0.6$

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EVIDENCE THEORY (DEMPSTER–SHAFER)

Relationship to traditional probability

- subjective bounding property for given basic probability assignment
  $\mathrm{Bel}(.) \leq \mathrm{P}(.) \leq \mathrm{Pl}(.)$
- generation schemes for observations not considered
- subjective character of the basic probability assignment
- statistical methods (estimations and tests) not defined
- covers the special case of traditional probability
  » focal subsets are disjoint singletons (dissonant case), as images of elementary events
  → $\mathrm{Bel}(.) = \mathrm{P}(.) = \mathrm{Pl}(.)$

Information content

- possible value ranges with crisp bounds for a series of observed events
- bounds on subjective probability

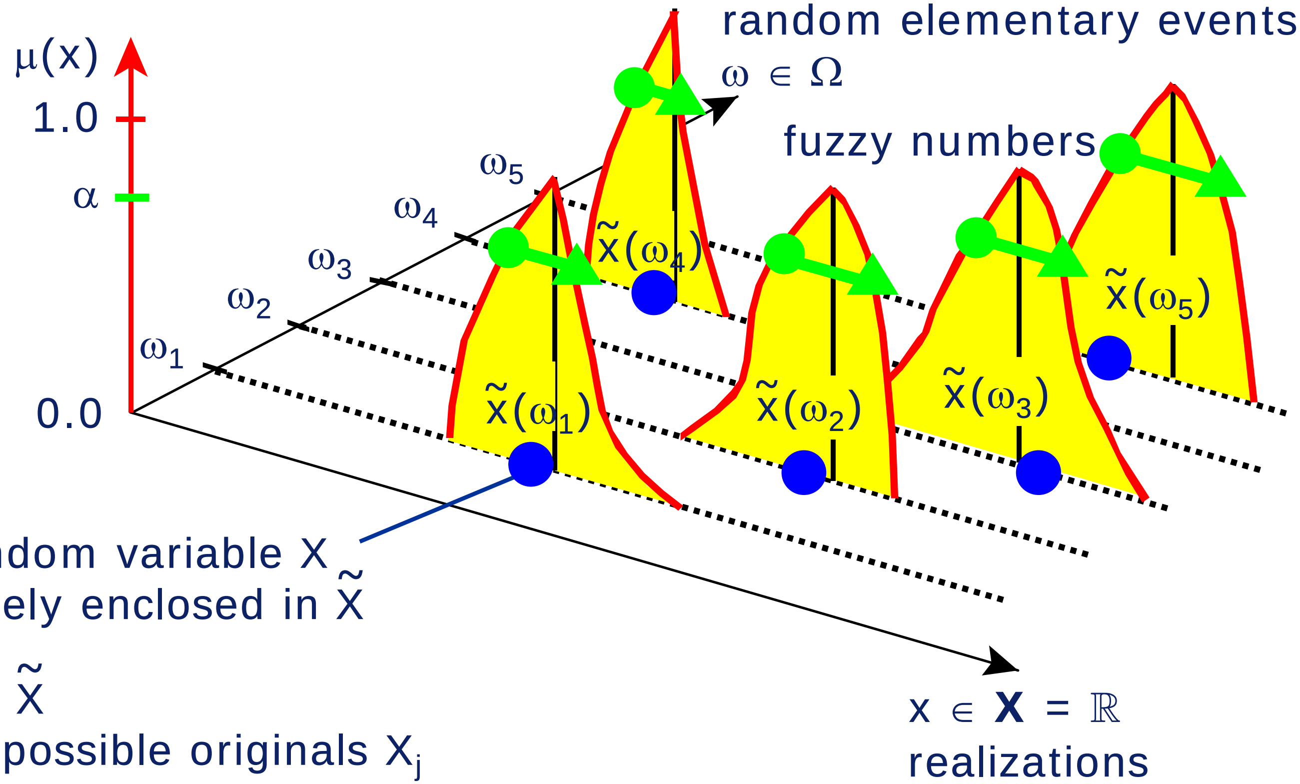
Numerical processing

- direct processing of empirical information (no theoretical distribution model)
- stochastic simulation (Monte Carlo, Latin Hypercube)
  → general applicability

FUZZY RANDOM VARIABLES

Mathematical model

- fuzzy realizations generated by elementary events
  $\tilde{X}: \Omega \rightarrow \mathbb{F}(\mathbb{R})$



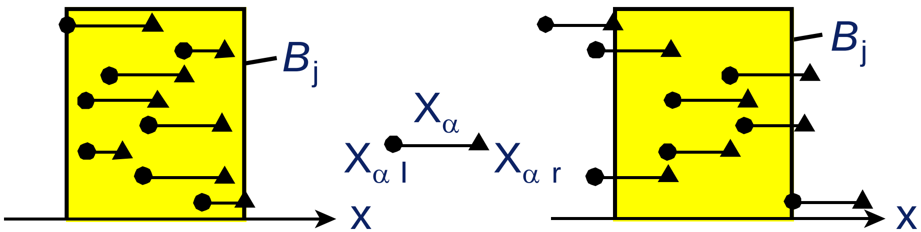
- original X_j
  » real-valued random variable X that is completely enclosed in $\tilde{X}$
- representation of $\tilde{X}$
  » fuzzy set of all possible originals X_j
- α-discretization → random α-level sets X_α

FUZZY PROBABILITIES

Mathematical model (cont'd)

- fuzzy probability
  - » $\tilde{P}(B_j) = \{(P_\alpha(B_j),\ \mu(P_\alpha(B_j)))\},\ P_\alpha(B_j) = [P_{\alpha l}(B_j), P_{\alpha r}(B_j)],\ \mu(P_\alpha(B_j)) = \alpha\ \forall\alpha \in (0,1]$
  - » evaluation of all random α-level sets X_α

    $P_{\alpha l}(B_j) = P(X_\alpha \subseteq B_j)$ $\qquad$ $P_{\alpha r}(B_j) = P(X_\alpha \cap B_j \neq \varnothing)$

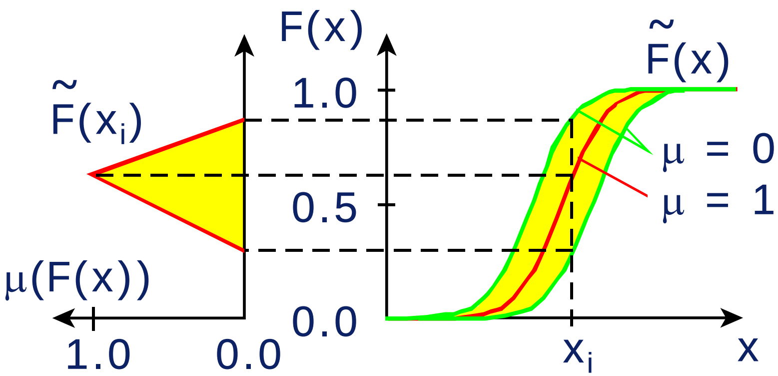


note:
in evidence theory,
these events represent
"belief" and "plausibility"

- fuzzy probability distribution functions
  - » bunch of the distribution functions $F_j(x)$ of the originals X_j of $\tilde{X}$

➡ $\tilde{F}(x)$ with fuzzy parameters and fuzzy functional type

(related to p-box approach for α = 0)



FUZZY PROBABILITIES

Relationship to evidence theory

- focal subsets represented by fuzzy realizations
- probability of the focal subsets induced by elementary events

Relationship to traditional probability

- traditional probabilistic generation scheme for realizations
- statistical methods (estimations and tests) fully applicable to originals
- fuzzy set of possible probabilistic models ➡ subjective bounding property
- special case of traditional probability directly included with $P_{\mu = 1}(.)$

Information content

- set of probabilistic models with subjective weights for their plausibility
- probability bounds for various intensities of imprecision in the model

Numerical processing

- utilization of the complete framework of stochastic simulation
- fuzzy / interval analysis applied to fuzzy parameters and probabilities
  ➡ general applicability

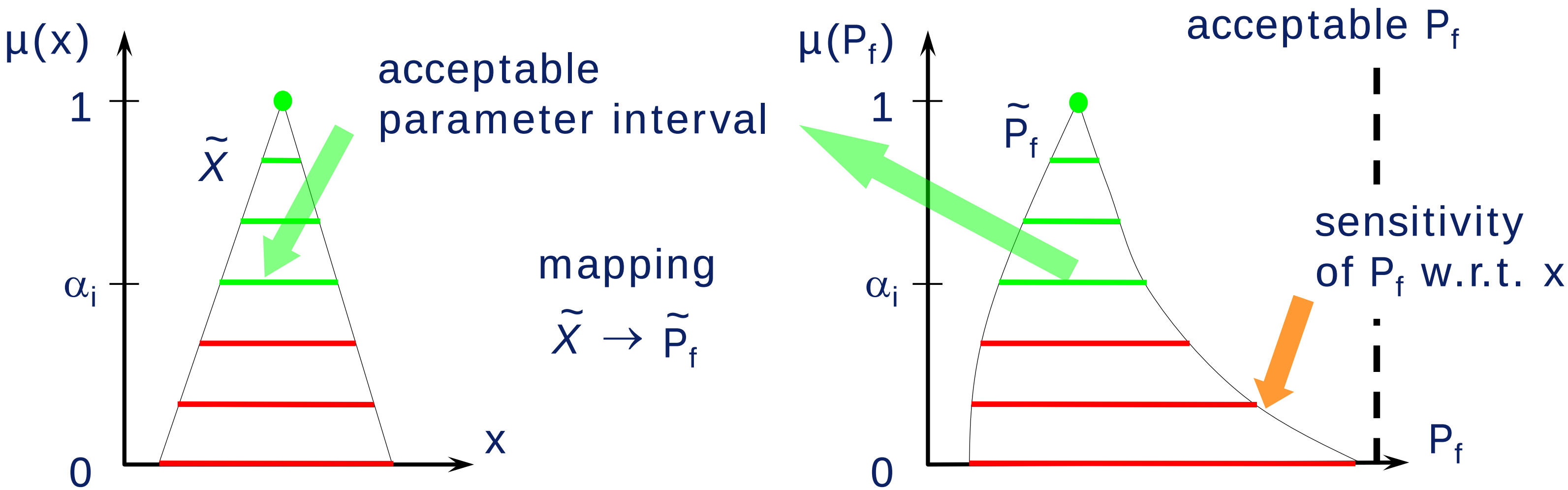
▪ sensitivities with respect to probabilistic model choice
▪ conclusions in association with the initial observation

FUZZY PROBABILITIES IN RELIABILITY ANALYSES

Fuzzy parameters

- structural parameters
- probabilistic model parameters

Failure probability

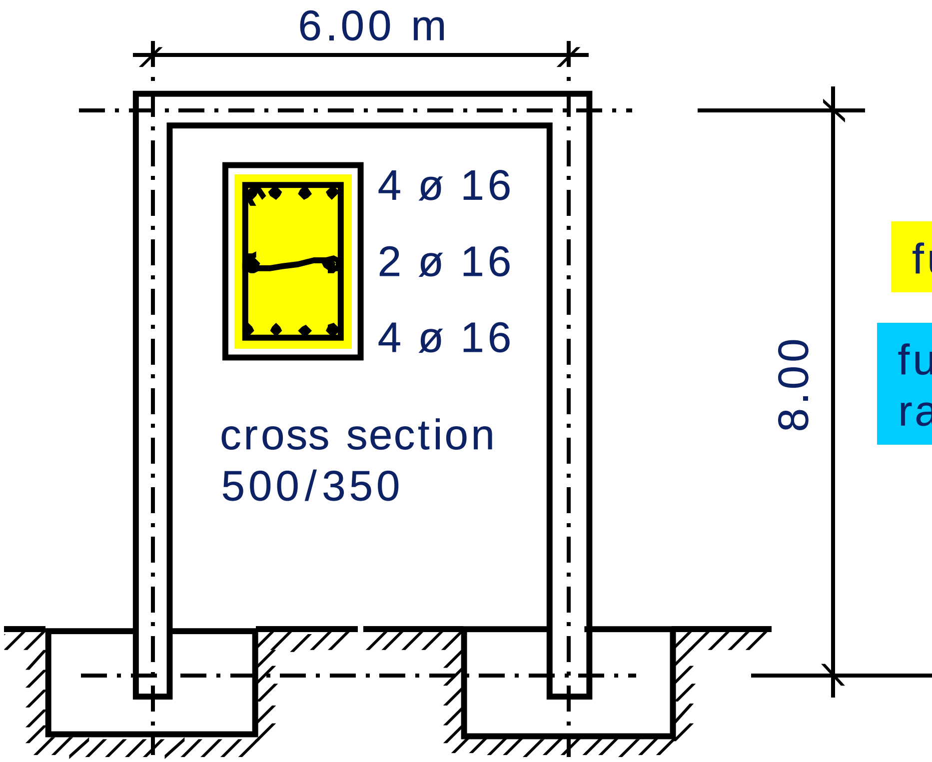


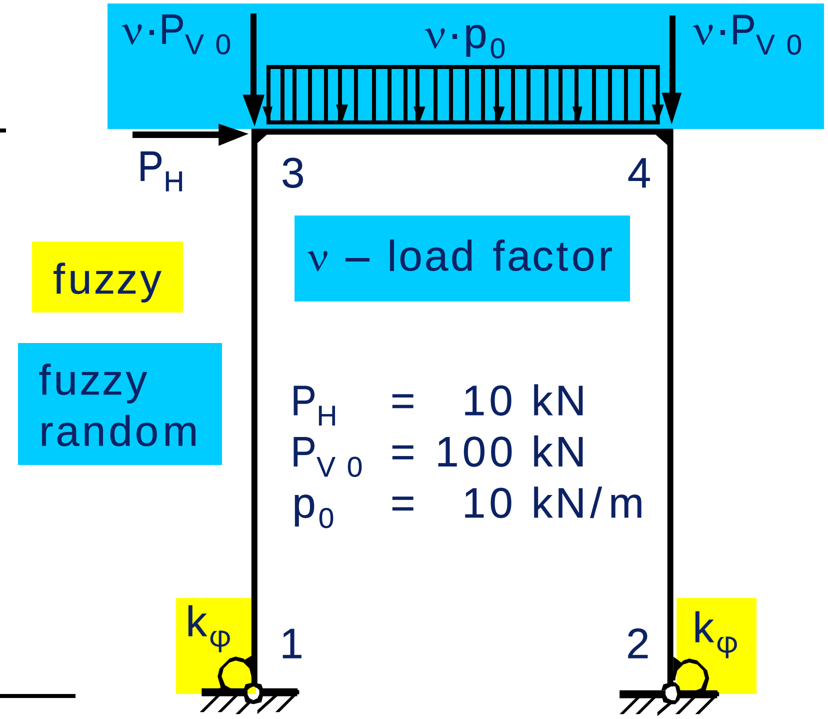
- ➡ coarse specifications of design parameters & probabilistic models
- ➡ attention to / exclude model options leading to large imprecision of P_f
- ➡ acceptable imprecision of parameters & probabilistic models
- ➡ indications to collect additional information
- ➡ definition of quality requirements
- ➡ robust design

μ(.) not important, but analysis with various intensities of imprecision

EXAMPLE 1 – RELIABILITY ANALYSIS

Reinforced concrete frame





Loading Process

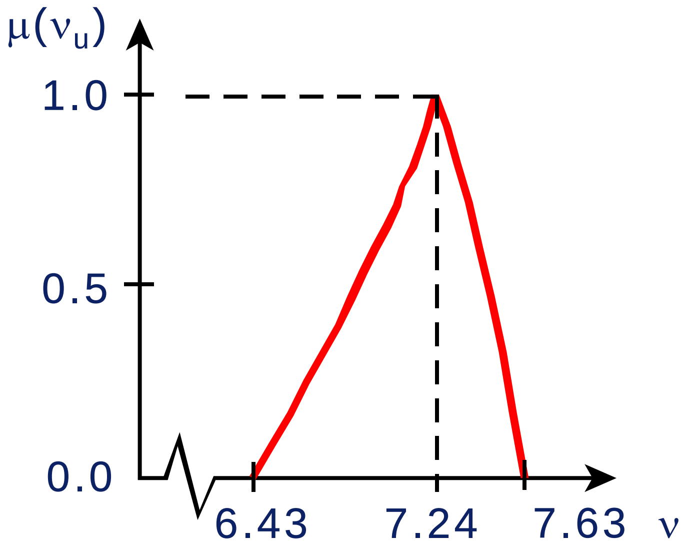
- dead load
- horizontal load P_H
- vertical loads $\nu \cdot P_{V\,0}$ and $\nu \cdot p_{V\,0}$

simultaneous processing of imprecision and uncertainty

EXAMPLE 1 – RELIABILITY ANALYSIS

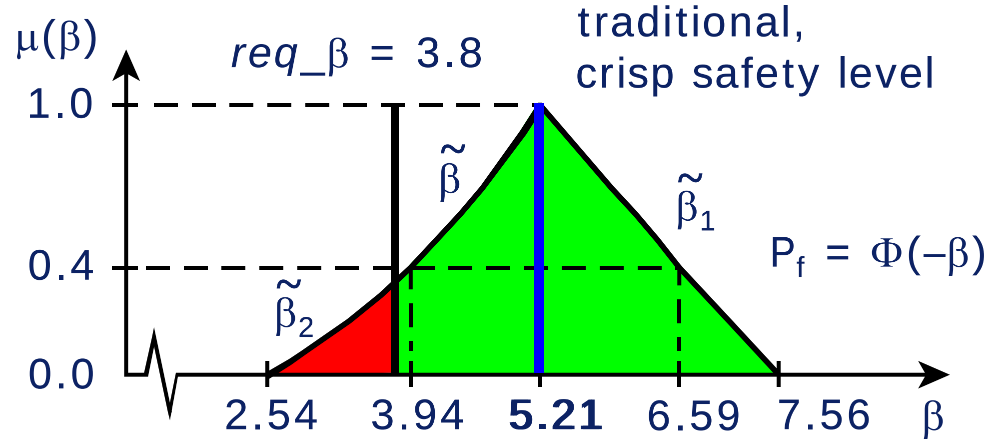
Structural response

- fuzzy failure load (limit state)



Safety Level

- fuzzy reliability index (fuzzy-FORM)



$\tilde{\beta}_2 < 3.8 = req\_\beta$ | $\tilde{\beta}_1 \geq 3.8 = req\_\beta$

worst and best case results for various intensities of imprecision

➡ maximum intensity of imprecision to meet requirements

➡ required reduction of input imprecision

EXAMPLE 2 – RELIABILITY ANALYSIS

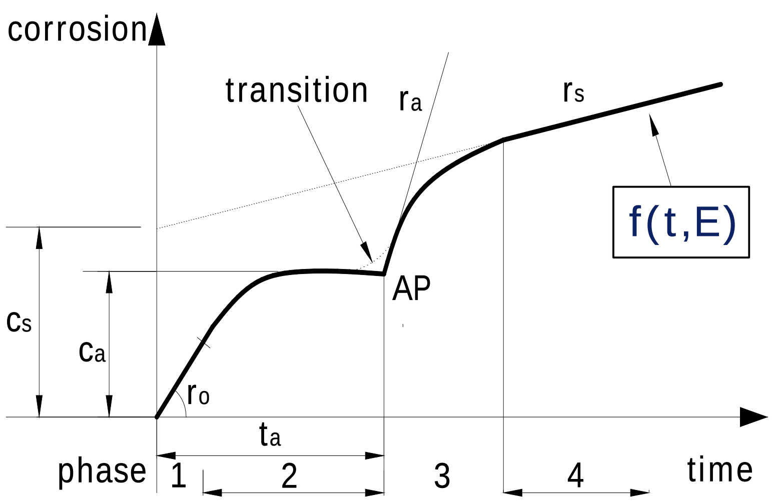
Fixed jacket platform

- imprecision in the models for
  - » wave, drag and ice loads
  - » wind load
  - » corrosion
  - » joints of tubular members
  - » foundation
  - » possible damage

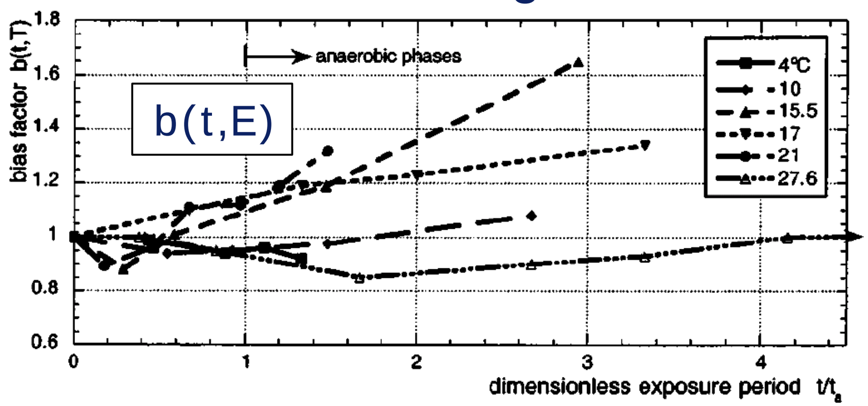
EXAMPLE 2 – RELIABILITY ANALYSIS

Probabilistic model *(After R.E. Melchers)*

- $c(t,E) = b(t,E) \cdot f(t,E) + \varepsilon(t,E)$ (time-dependent corrosion depth, uniform)
  - » c(t,E) – corrosion depth
  - » f(t,E) – mean value function
  - » b(t,E) – bias function
  - » ε(t,E) – uncertainty function (zero mean Gaussian white noise)
  - » E – collection of environmental and material parameters



Modeling ?



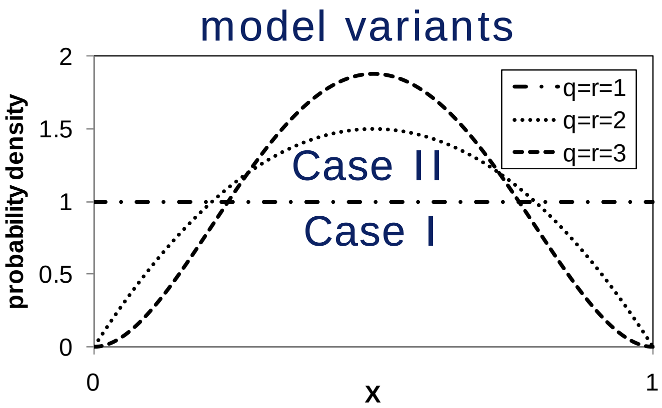
EXAMPLE 2 – RELIABILITY ANALYSIS

Models for the bias factor

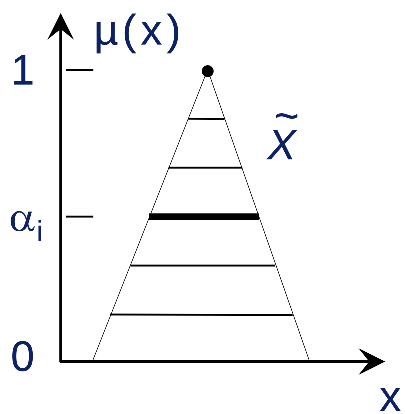
- Bounded random variable – beta distribution

$$f_X(x) = \frac{1}{B(q,r)} \cdot \frac{(x-a)^{q-1} \cdot (b-x)^{r-1}}{(b-a)^{q+r-1}}, \quad a \leq x \leq b$$

with X representing a random bias b(t,E)



- interval and fuzzy set



Reliability analysis

- select one value b(t,E) = x
- calculate P_f via Monte Carlo simulation
  - » beta distribution ➡ pdf for P_f ("overall" sensitivity of P_f)
  - » interval ➡ interval for P_f (bounds)
  - » fuzzy set ➡ fuzzy set for P_f (set of intervals with various intensities of imprecision ➡ "incremental" sensitivities of P_f)

EXAMPLE 2 – RELIABILITY ANALYSIS

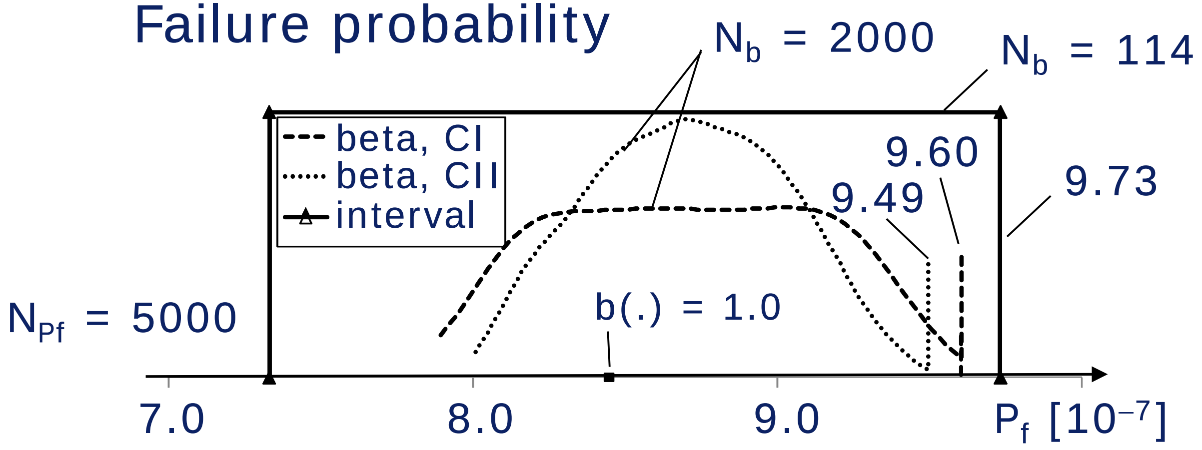
Fixed jacket platform

- dimensions
  - » height: 142 m
  - » top: 27 X 54 m
  - » bottom: 56 X 70 m
- loads, environment
  - » T = 15°C, t = 5 a
  - » random: wave height, current, yield stress, and corrosion depth c(t,E)
  - » beta distribution / interval for b(.) ∈ [0.8,1.6]

Reliability analysis

- Monte Carlo simulation with importance sampling and response surface approximation

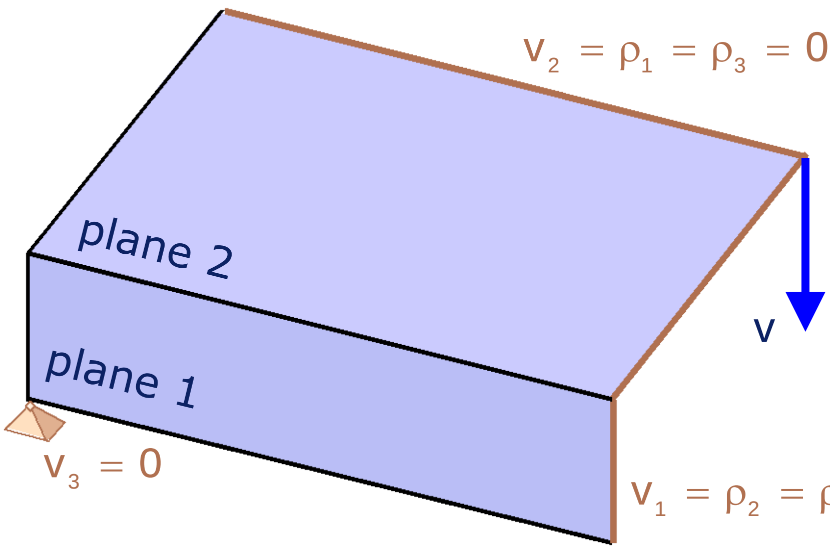
Failure probability

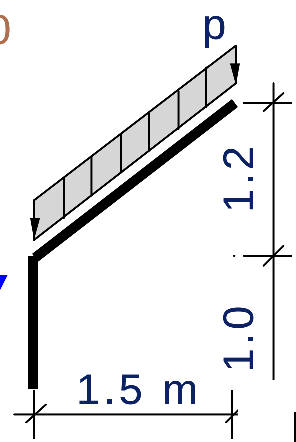


EXAMPLE 3 – STRUCTURAL ANALYSIS *(DFG SFB 528)*

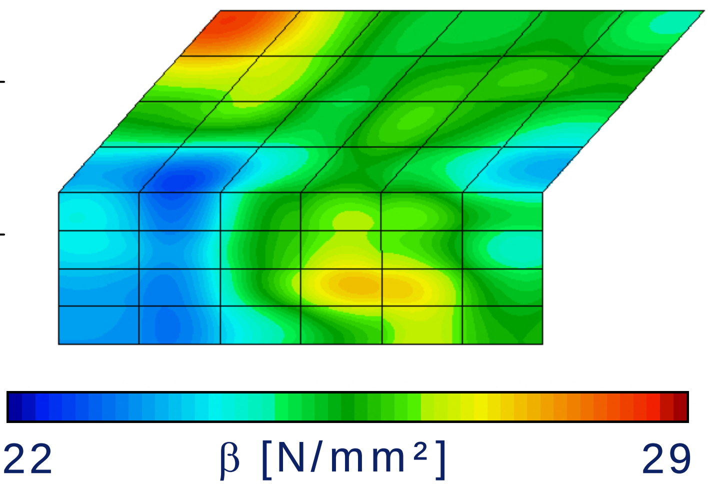
Textile reinforced roof structure

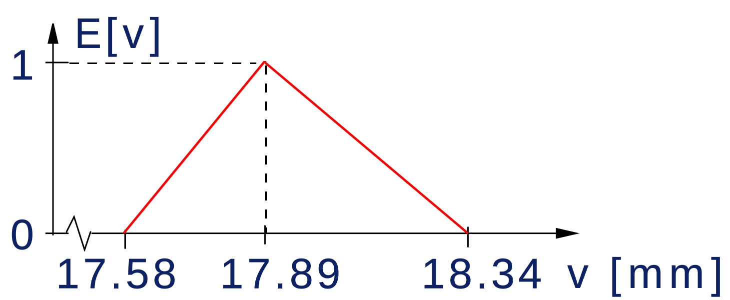
- stochastic finite elements

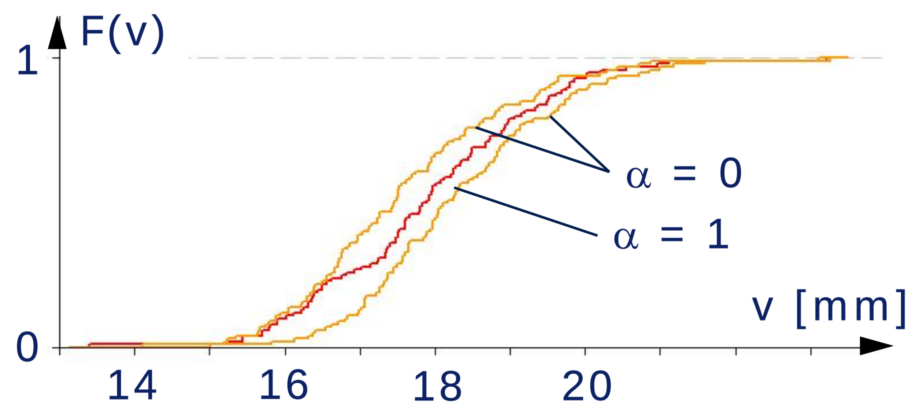




- fuzzy random field for material strength



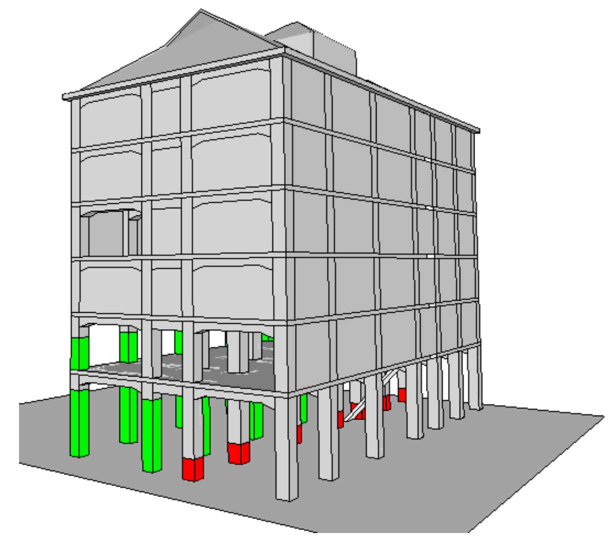




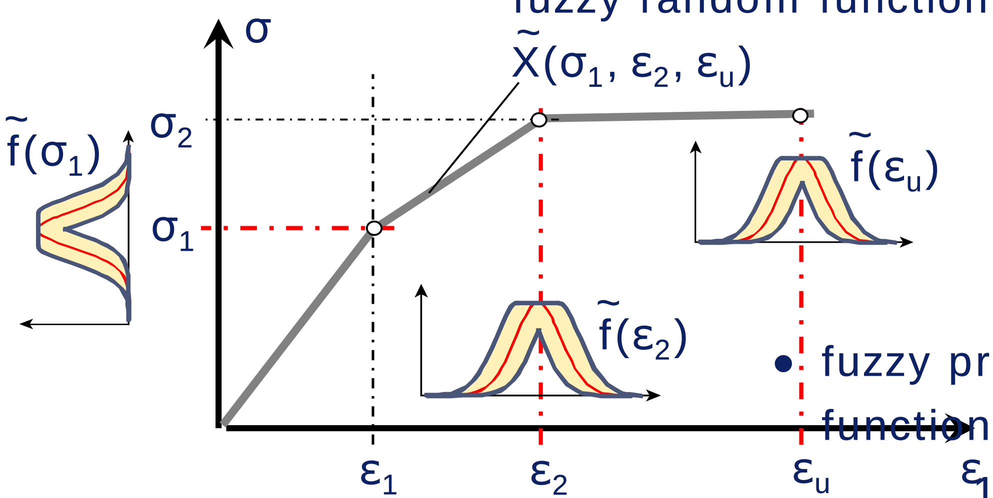
EXAMPLE 4 – COLLAPSE SIMULATION

(DFG Research Unit 500)

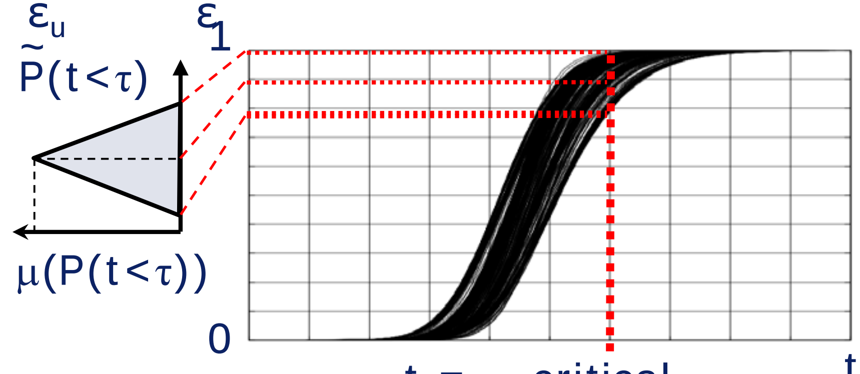
Controlled demolition of a store house



- fuzzy random variables
  - » material behavior



- fuzzy probability distribution function $\tilde{F}(t)$ for failure time



- fuzzy stochastic structural analysis
  - » Monte Carlo simulation with response surface approximation

RESUMÉ

Concepts of imprecise probabilities

- high degree of flexibility in uncertainty quantification
  - » modeling adjustable to the particular situation
  - » inclusion of vague and imprecise information
  - » utilization of traditional stochastic methods and techniques
- comprehensive reflection of imprecision in the computational results; bounds on probability
  - » aspects in safety and design
    - ▪ identification of sensitivities, hazards and robust design
    - ▪ dealing with coarse specifications in early design stages
  - » economic aspects
    - ▪ reduction of required information: only as much and precise as necessary
    - ▪ re-assessment of existing, damaged structures: quick first assessment and justification of further detailed investigation

➡ appropriate model choice in each particular case
depending on the available information AND the purpose of the analysis

➡ application of different uncertainty models in parallel