Interval quasi-Monte Carlo method for reliability assessment with imprecise probability

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Abstract. Reliability analysis of structures is often performed on the basis of limited data. Under this circumstance, there are practical difficulties in identifying unique distributions as input for a probabilistic analysis. However, the selection of realistic probabilistic input is critical for the quality of the results of the reliability analysis. This problem can be addressed by using an entire set of plausible distribution functions rather than one single distribution for random variables based on limited data. An imprecise probability distribution can be modeled by a probability box, i.e., the bounds of the cumulative distribution function for the random variable. Sampling-based methods have been proposed to perform reliability analysis with probability boxes. Direct sampling of probability boxes requires a large number of samples. The computational cost can be very high as each simulation involves an interval analysis. This study proposes an interval quasi-Monte Carlo simulation methodology to efficiently compute the bounds of structure failure probabilities. The methodology is based on deterministic low-discrepancy sequences, which are distributed more regularly than the (pseudo) random points in direct Monte Carlo simulation. The efficiency and accuracy of the present method is illustrated using two examples. The reliability implications of different approaches for construction of probability boxes are also investigated through the example.

Keywords: epistemic uncertainty, imprecise probability, interval, low-discrepancy sequence, probability box, quasi-Monte Carlo, structural reliability.

1. Introduction

The building process of civil engineering structures and infrastructure is complicated by the various sources of uncertainties in structural resistance and loads, as well as in computational models. These uncertainties are treated as random variables when using established probabilistic methods for reliability analysis. Structural reliability is then measured by a probability of failure, denoted P_f (Melchers, 1999).

Although the mathematical formulation as well as the basic numerical techniques for the calculation of P_f appears to be straightforward, difficulties appear in practical applications. For one thing, the evaluation of the involved multi-dimensional integral can be challenging. Development of numerical techniques target at a high numerical efficiency and concern advanced concepts of Monte Carlo (MC) simulation such as subset simulation (Au and Beck, 2001) and line sampling (Koutsourelakis et al., 2004). Particular attention is currently paid to simulation schemes for high-dimensional problems. Moreover, there are practical difficulties in identifying the proper distribution for the random variables particularly to model their extremes

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("tails") which are of greatest concern in reliability assessment. Structural failure probabilities, generally very small, are sensitive to the choice of probability distributions (Ellingwood, 2001). However, available data on structural strength and loads are typically limited, and competing distributions often cannot be distinguished by standard statistical tests. When observational data are limited, the analyst may not be able to identify the type of the distribution of a random variable, or precise values for the distribution parameters, or there may be competing probabilistic models. The selection of a distribution for the probabilistic input is so generally realized based on ambiguous information and indications. This may lead to a wrong model choice and a strong overestimation of structural reliability resulting in critical decisions with severe consequences.

It is advisable to consider the distribution itself as uncertain. These statistical uncertainties are epistemic (knowledge-based) in nature (Der Kiureghian and Ditlevsen, 2009). Because of the distribution sensitivity problem, the failure probability calculated on the basis of small data samples is only a "notional" value and should not be interpreted as a relative frequency in an absolute sense (Ellingwood, 2001; Der Kiureghian and Ditlevsen, 2009). To overcome the distribution sensitivity problem, the development of first-generation probability-based structural design codes utilized a process of calibration to existing practice. The notional reliabilities associated with existing and new practices were computed in the same model universe of probability distributions and used as a means of comparison (Ellingwood et al., 1982). If the calculated reliabilities are notional and only used for the purpose of reliability comparison, then the uncertainty in the distributions is generally of minor importance and can be ignored. A reliability comparison in this sense represents a ranking of alternatives, whereby the uncertainties in the distributions associated with the alternatives are normally not significantly different in magnitude since they origin from the same source (same model universe of probability distributions). The effects of the uncertainties in the distributions, hence, cancel out one another on an ordinal (ranking) scale almost completely.

However, there are circumstances where epistemic uncertainties due to limited availability of data need to be included explicitly in reliability analysis and, further on in risk assessment. One such case is the risk-informed decision-making, in which the regulatory authorities often see a need to quantify the confidence in the results of the risk assessment, particularly if the event is rare but the consequence is severe. Another case is the performance-based approach to structural design. In this new paradigm of structural engineering, it is necessary to establish explicit reliability/risk terms to rationalize the selection of performance levels in structural design. Designers using innovative building materials and technologies are also concerned with computing realistic structural reliabilities because they cannot rely on past experience to calibrate the reliabilities. In all these circumstances a notional reliability measure without taking into account the epistemic uncertainty in the distributions is not very helpful.

Within a pure probabilistic framework, epistemic uncertainty can be handled with Bayesian approaches. Uncertain parameters of a probabilistic model can be described with prior distributions and updated by means of limited data. They can then be introduced formally, with the remaining (aleatory) uncertainties, in the reliability calculation (Der Kiureghian and Ditlevsen, 2009; Ellingwood and Kinali, 2009). In the case where competing probabilistic models exist, each model is considered separately with an assigned probability mass. A failure probability can be computed for each probabilistic model. The expectation of the failure probability can then be calculated as a characteristic result, and the frequency distribution (or variance) of the failure probability can be evaluated to separate the effects of aleatory and epistemic uncertainty. In practical applications, this requires a high numerical effort or statistical approximations.

Alternatively, an imprecisely known probability distribution can be modeled by a family of all candidate probability distributions which are compatible with available data. This is the idea of the theory of imprecise

probabilities (Walley, 1991). Dealing with a set of probability distributions is essentially different from a Bayesian approach. A practical way to represent the distribution family is to use a probability bounding approach by specifying the lower and upper bounds of the imprecise probability distribution. This corresponds to the use of an interval to represent an unknown but bounded number. Consequently, a unique failure probability cannot be determined. Instead, the failure probability is obtained as an interval whose width reflects the imprecision of the distribution model in the calculated reliability.

A popular uncertainty model using the probability bounding approach is the probability box (p-box for short) structure (Ferson et al., 2003). A p-box is closely related to other set-based uncertainty models such as random sets, Dempster-Shafer evidence theory and random intervals. In many cases, these uncertainty models can be converted into each other, and thus considered to be equivalent (Walley, 2000; Ferson et al., 2003; Joslyn and Ferson, 2004; Möller and Beer, 2008; Baudrit et al., 2008). Therefore, the methodology presented in this paper is also applicable to other set-based uncertainty models.

Within the reliability analysis context, simulation-based methods have been suggested to propagate p-boxes (Alvarez, 2006; Baraldi et al., 2008; Batarseh and Wang, 2008; Zhang et al., 2010a). Direct sampling of p-boxes requires a large number of samples to control the sampling uncertainty. The total computational cost can be very high as each simulation may involve an expensive range-finding problem. The issue of computational cost becomes more serious when the limit state function is only implicitly known through a computational model. Another concept follows the idea of global optimization to directly identify the bounds of probabilistic results (Möller and Beer, 2004). Although this concept is general and can be applied in association with variance-reduction methods (Zhang et al., 2010b), the computations are still numerically demanding. There is some urgency for further developments towards efficient methodologies for analysis with imprecise probabilities.

This study focuses on the reduction of sampling uncertainty with quasi-Monte Carlo technique. Quasi-MC method is typically used for multidimensional numerical integration problems. It performs in a similar manner as the Monte Carlo integration with the exception that samples used are a systematically constructed deterministic sequence instead of random samples. This deterministic sequence, known as low-discrepancy sequence, has the property that its points approximate the uniform distribution as closely as possible, so that a better sampling of the function can be achieved (Lemieux, 2009). As compared to MC method, Quasi-MC method can often achieve a desired tolerance of the result variance using much less simulation effort. In addition, the implementation of quasi-MC is as simple as replacing the pseudo-random numbers in MC method by low-discrepancy sequences.

Quasi-Monte Carlo has been traditionally used in computational finance and computational physics. Its applications in structural reliability analysis have been more recent (Nie and Ellingwood, 2004; Shinoda, 2007; Dai and Wang, 2009). In this paper the quasi-Monte Carlo technique is combined with the interval sampling to develop a simple, yet efficient sampling procedure for reliability analysis with p-boxes. Variance-type error estimates for the proposed method are calculated. Different approaches for constructing p-boxes are investigated and compared.

2. Construction of probability box

An imprecise distribution can be represented by the bounds on its cumulative distribution function (CDF). Let $F_X(x)$ denote the CDF for a random variable X. With an imprecise distribution, for any reference

number x it is assumed that a closed interval $[\underline{F}_{\mathrm{X}}(x),\ \overline{F}_{\mathrm{X}}(x)]$ can be found to bound the possible values of $F_{\mathrm{X}}(x)$, i.e., $\underline{F}_{\mathrm{X}}(x) \leq F_{\mathrm{X}}(x) \leq \overline{F}_{\mathrm{X}}(x)$. $\underline{F}_{\mathrm{X}}(x)$ and $\overline{F}_{\mathrm{X}}(x)$ thus form the envelopes of the probability family

$$\mathscr{P} = \{P | \forall x \in \mathbb{R}, \underline{F}_{\mathrm{X}}(x) \leq F_{\mathrm{X}}(x) \leq \overline{F}_{\mathrm{X}}(x)\}. \tag{1}$$

Such a pair of two CDF's $\underline{F}_{\mathrm{X}}(x)$ and $\overline{F}_{\mathrm{X}}(x)$ defines a "probability box". Detailed background can be found in (Ferson et al., 2003).

Various ways have been suggested to construct bounds on cumulative distribution functions. Depending on the amount of available information, the analyst may choose among the following options to construct a p-box.

2.1. KOLMOGOROV-SMIRNOV (K-S) CONFIDENCE LIMITS

K-S confidence limits on an empirical distribution have been proposed in (Ferson et al., 2003) to define bounds of the CDF as

$$\begin{aligned} \underline{F}_{\mathrm{X}}(x) &= \min(1, \max(0, F_n(x) - D_n^{\alpha})), \\ \overline{F}_{\mathrm{X}}(x) &= \min(1, \max(0, F_n(x) + D_n^{\alpha})), \end{aligned} \tag{2}$$

in which $F_n(x)$ is the empirical cumulative frequency function, and D_n^{α} is the K-S critical value at significance level α with a sample size of n.

2.2. CHEBYSHEV'S INEQUALITY

If only knowledge of the first two moments (μ and σ) for the random variable is available, Chebyshev's inequality is proposed in (Oberguggenberger and Fellin, 2008) to define a p-box as

$$\begin{aligned} \underline{F}_{\mathrm{X}}(x) &= \begin{cases} 0, & x < \mu - \sigma, \\ 1 - \frac{\sigma^2}{(x-\mu)^2}, & x \geq \mu + \sigma, \end{cases} \\ \overline{F}_{\mathrm{X}}(x) &= \begin{cases} \frac{\sigma^2}{(x-\mu)^2}, & x < \mu - \sigma, \\ 1, & x \geq \mu - \sigma. \end{cases} \end{aligned} \tag{3}$$

2.3. DISTRIBUTIONS WITH INTERVAL PARAMETERS

Probability families can also be defined by distributions with interval parameters as described in (Zhang et al., 2010a). This concept corresponds to the bunch parameter representation of fuzzy random variables described in (Möller and Beer, 2004). Confidence intervals on statistics (e.g., mean, variance) provide a natural way to define interval bounds of the distribution parameters. Based on the observational data, a confidence interval can be established using classical statistical approaches, such that the (unknown) distribution parameter is located within the interval at a specified level of confidence. Let Θ denote the (unknown) statistical parameter of the distribution, and its interval range is $[\underline{\theta},\ \overline{\theta}]$, thus the p-box can be constructed as

$$\begin{aligned} \underline{F}_{\mathrm{X}}(x) &= \min\{F_{\mathrm{X}}(x|\theta) : \underline{\theta} \leq \theta \leq \overline{\theta}\}, \\ \overline{F}_{\mathrm{X}}(x) &= \max\{F_{\mathrm{X}}(x|\theta) : \underline{\theta} \leq \theta \leq \overline{\theta}\}, \end{aligned} \tag{4}$$

in which $F_{\mathrm{X}}(x|\theta)$ is the conditional CDF, given that the parameter Θ takes the value of θ. A related concept for the quantification of fuzzy random variables can be found in (Beer, 2009).

2.4. Envelope of competing probabilistic models

When there are multiple candidate distribution models which cannot be distinguished by standard goodness-of-fit tests, the envelope of the candidate distributions may be used as the p-box (Fu et al., 2011), i.e.,

$$\begin{aligned} \underline{F}_{\mathrm{X}}(x) &= \min\{F_i(x),\ i = 1, \ldots\}, \\ \overline{F}_{\mathrm{X}}(x) &= \max\{F_i(x),\ i = 1, \ldots\}, \end{aligned} \tag{5}$$

where $F_i(x)$ denotes the ith candidate CDF.

2.5. Summary

One important observation about p-box is that although its construction still needs some subjective judgement, such as selecting an appropriate confidence level in the approach of K-S confidence limit, it generally requires less subjective information than the Bayesian approaches.

Among the above four approaches of deriving p-boxes, the K-S approach and Chebyshev's inequality are non-parametric and do not require a distribution assumption, in contrast to the third approach "distributions with interval parameters". The fourth approach "envelope of competing probabilistic models" may be parametric or non-parametric. As will be seen in the second example, the p-boxes obtained from these approaches may differ significantly from each other, leading to different results. This brings us to an important question: for a given set of small samples, how should we construct an appropriate p-box for reliability assessment? We will return to this question in our discussion of Example 2.

3. Direct interval Monte Carlo simulation

Reliability analysis with probability boxes can be performed using sampling-based methods (Alvarez, 2006; Baraldi et al., 2008; Batarseh and Wang, 2008; Zhang et al., 2010a). We start with the conventional MC simulation, in which the failure probability P_f is estimated by

$$P_f \approx \hat{P}_f = \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}[g(\mathbf{x}_j) \leq 0], \tag{6}$$

where n is the total number of simulations conducted, and $\mathbf{x}_j$ represents the jth simulated vector of basic random variables in accordance to their joint probability density function $f_{\mathbf{X}}(\,)$. $g(\,)$ is the limit state function and failure occurs when $g(\mathbf{X}) \leq 0$. $\mathbf{1}[\ ]$ is the indicator function, having the value 1 if $[\ ]$ is "true" and the value 0 if $[\ ]$ is "false".

As a basis for our development we use the inverse transform method (Lemieux, 2009) for generation of random numbers:

$$\mathbf{x}_j = F_{\mathrm{X}}^{-1}(\mathbf{r}_j), \quad j = 1, 2, \ldots, n \tag{7}$$

with $F_{\mathrm{X}}(\,)$ being the target CDF of X, and $\mathbf{r}_1, \ldots \mathbf{r}_n$ a sample of independent and identically distributed (i.i.d.) standard uniform random variates.

Now consider the situation in which $F_{\mathrm{X}}(\,)$ is unknown but bounded by $\underline{F}_{\mathrm{X}}(\,)$ and $\overline{F}_{\mathrm{X}}(\,)$. Since only bounds of $F_{\mathrm{X}}(\,)$ are known, it is not possible to generate point samples but only interval samples. Let

$$\begin{aligned} \underline{\mathbf{x}}_j &= \overline{F}_X^{-1}(\mathbf{r}_j), \\ \overline{\mathbf{x}}_j &= \underline{F}_X^{-1}(\mathbf{r}_j), \quad j = 1, \ldots, n, \end{aligned} \tag{8}$$

it follows that $\underline{\mathbf{x}}_j \leq \mathbf{x}_j \leq \overline{\mathbf{x}}_j$. Note that the inverse of a CDF, $F_X^{-1}(\,)$, is monotonically decreasing. Eq. (8) suggests that for a given $\mathbf{r}_j$, the interval $[\underline{\mathbf{x}}_j, \overline{\mathbf{x}}_j]$ contains all possible simulated numbers from the probability family represented by the p-box. Define

$$\begin{aligned} \underline{g}_j(\mathbf{x}) &= \min\{g(\mathbf{X}) : \underline{\mathbf{x}}_j \leq \mathbf{X} \leq \overline{\mathbf{x}}_j\}, \\ \overline{g}_j(\mathbf{x}) &= \max\{g(\mathbf{X}) : \underline{\mathbf{x}}_j \leq \mathbf{X} \leq \overline{\mathbf{x}}_j\}, \end{aligned} \tag{9}$$

one has

$$\mathbf{1}[\overline{g}_j(\mathbf{x}) \leq 0] \leq \mathbf{1}[g(\mathbf{x}_j) \leq 0] \leq \mathbf{1}[\underline{g}_j(\mathbf{x}) \leq 0]. \tag{10}$$

Applying the inequality (10) to Eq. (6) yields

$$\frac{1}{n}\sum_{j=1}^{n} \mathbf{1}[\overline{g}_j(\mathbf{x}) \leq 0] \leq P_f \leq \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}[\underline{g}_j(\mathbf{x}) \leq 0]. \tag{11}$$

Therefore, a lower and an upper bound for P_f can be estimated as:

$$\begin{aligned} \underline{P}_f &= \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}[\overline{g}_j(\mathbf{x}) \leq 0], \\ \overline{P}_f &= \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}[\underline{g}_j(\mathbf{x}) \leq 0]. \end{aligned} \tag{12}$$

Because the randomly generated samples are intervals, this procedure is referred to as direct interval Monte Carlo simulation (Zhang et al., 2010a).

3.1. Variance of direct interval Monte Carlo

The accuracy of the reliability bounds as obtained from Eq. (12) can be measured by their variance. An estimator for the variance of $\underline{P}_f$, denoted $\widehat{\mathrm{Var}}(\underline{P}_f)$, is constructed by calculating its sample variance

$$\widehat{\mathrm{Var}}(\underline{P}_f) = \tfrac{1}{n(n-1)}\left(\textstyle\sum_{j=1}^{n} \mathbf{1}^2[\overline{g}_j \leq 0] - \underline{P}_f^2\right) = \tfrac{\underline{P}_f - \underline{P}_f^2}{n-1}. \tag{13}$$

Similarly, the variance of $\overline{P}_f$ can be estimated as

$$\widehat{\mathrm{Var}}(\overline{P}_f) = \tfrac{1}{n(n-1)}\left(\textstyle\sum_{j=1}^{n} \mathbf{1}^2[\underline{g}_j \leq 0] - \overline{P}_f^2\right) = \tfrac{\overline{P}_f - \overline{P}_f^2}{n-1}. \tag{14}$$

4. Interval quasi-Monte Carlo simulation

The starting point of our development is the difficulty to implement ideally uniform random variates in the simulation procedure as described in Section 3. The accuracy of Monte Carlo sampling relies on the "uniformity" (equidistribution) of the used uniform random numbers, i.e., the better the samples approximate the uniform distribution the more precise the Monte Carlo result is (Kalos and Whitlock, 1986). In practice, pseudo uniform random numbers which are produced by deterministic algorithms are used. Unfortunately, the pseudo uniform random numbers commonly have poor equidistribution properties, and accordingly, a good result is only achievable, if possible at all, with a large number of samples. Hence, the computational cost of interval Monte Carlo simulations can be very high as each simulation may involve an expensive range-finding problem (i.e., Eq. 9).

4.1. Low-discrepancy sequences

Consider a general multivariate integral

$$I(h) = \int_{C^s} h(\mathbf{u}) d\mathbf{u}, \tag{15}$$

where $\mathbf{u} = (u_1, \ldots, u_s)$ is an s-dimensional vector in the hypercube $C^s = [0,1)^s$, and $h : C^s \to \mathbb{R}$ is a real-valued function. Note that, in general, the integration problem characterizing the reliability assessment can always be reformulated as an equivalent one with the integration domain defined over the unit hypercube C^s (Lemieux, 2009). Given some sample points $P_n = \{\mathbf{u}_j, j = 1, \ldots, n\} \subset C^s$, an estimator Q_n for the integral $I(h)$ can be calculated as

$$I(h) \approx Q_n = \frac{1}{n} \sum_{j=1}^{n} h(\mathbf{u}_j). \tag{16}$$

With Monte Carlo method, the points $\mathbf{u}_j$ are i.i.d. samples from the uniform distribution in C^s. Alternatively, deterministic low-discrepancy sequences can be used as $\mathbf{u}_j$ to improve sampling efficiency. The theoretical background of this can be seen from the Koksma-Hlawka inequality (as discussed in Niederreiter, 1992), which gives the upper bound on the absolute error of Q_n:

$$|I(h) - Q_n| \leq V(h) D_n^{(s)} \tag{17}$$

where $V(h)$ is the bounded total variation of function h over C^s in the sense of Hardy and Krause (see Niederreiter, 1992). The term $D_n^{(s)}$ is defined as

$$D_n^{(s)} = \sup_{\mathbf{u} \in C^s} |F_n(\mathbf{u}) - F(\mathbf{u})|, \tag{18}$$

in which $\mathbf{u} = \{u_1, \ldots, u_s\} \in C^s$, $F(\mathbf{u})$ is the CDF of the uniform distribution in C^s, i.e.,

$$F(\mathbf{u}) = \prod_{i=1}^{s} u_i, \tag{19}$$

and $F_n(\mathbf{u})$ represents the empirical CDF induced by the sample points P_n, i.e.,

$$F_n(\mathbf{u}) = \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}(\mathbf{u}_j \leq \mathbf{u}). \tag{20}$$

It may be observed that $D_n^{(s)}$ is actually the Kolmogorov-Smirnov statistics. In analytical number theory, $D_n^{(s)}$ is known as the *discrepancy* of the point set P_n. $D_n^{(s)}$ can be thought of as a measure of the uniformity of P_n. The inequality (17) suggests that the absolute error of the integration estimator is bounded and dominated by $D_n^{(s)}$ as $V(h)$ is a constant as far as h is given. Therefore, the quality of the estimator Q_n depends only on the discrepancy $D_n^{(s)}$, and the sample points with the smallest discrepancy (i.e., the so-called *low-discrepancy* sample) are the optimal sampling points in this sense. Since the low-discrepancy sequences are deterministic, simulation with low-discrepancy sequences is often referred to as quasi-Monte Carlo simulation.

It is possible to construct low-discrepancy sequence points for a given dimension (s) and number of samples (n) such that the deterministic error bound for Q_n is in $O(n^{-1}(\log(n))^s)$ (Fang and Wang, 1994). This is a substantial improvement of the direct Monte Carlo convergence rate of $O(n^{-1/2})$, particularly when the dimension s is not too large. The most commonly used low-discrepancy point sets and sequences include good lattice points (GLP), Halton sequence, Hammersley sequence, Sobol' sequence, Faure sequence, etc. Among them GLP is a type of lattice points, and the others are digital sequences. Dai and Wang (2009) have compared the efficiency of various digital sequences and GLP in the context of structural reliability assessment. This study uses GLP, Halton, and Faure sequences. The construction of these point sets is presented in Appendix. Details can be found, for example, in (Niederreiter, 1992) and (Lemieux, 2009).

The uniformity of a low-discrepancy sequence may be visualized by plotting its two-dimensional projection (Morokoff and Caflisch, 1994). Fig. 1 shows two-dimensional scatter plots for random sampling, GLP, Halton and Faure sequences. It is evident that the low-discrepancy points fill in C^2 more uniformly than the pure random samples.

4.2. Interval quasi-Monte Carlo sampling procedure

The proposed interval quasi-Monte Carlo method performs in the same manner as the direct interval Monte Carlo method, except that the random uniform variates are replaced by deterministic low-discrepancy sequences. With the interval quasi-Monte Carlo method, the bounds of failure probability are given by:

$$\begin{aligned} \underline{P}_f &= \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}[\overline{g}_j(\mathbf{u}_j) \le 0], \\ \overline{P}_f &= \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}[\underline{g}_j(\mathbf{u}_j) \le 0], \end{aligned} \tag{21}$$

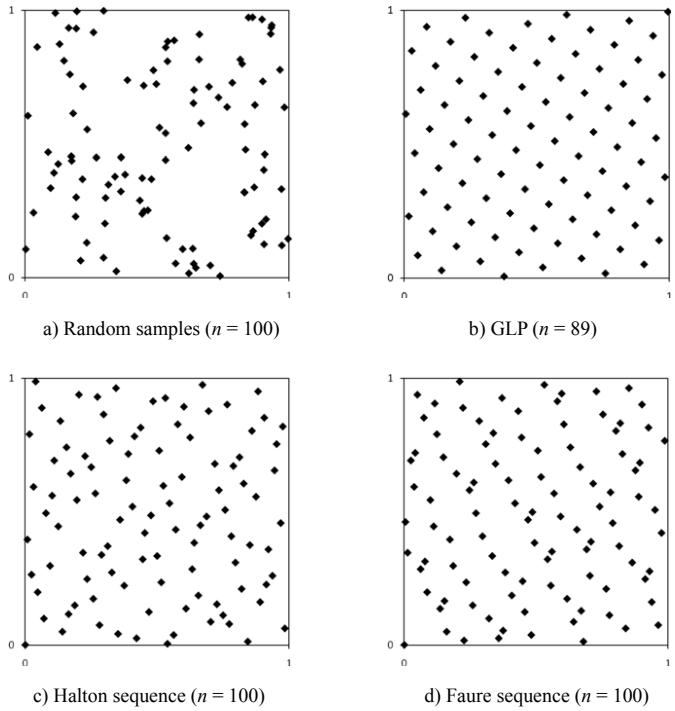
in which the sequence $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a low-discrepancy point set, and

$$\begin{aligned} \underline{g}_j(\mathbf{u}_j) &= \min\{g(\mathbf{X}) : \overline{F}_X^{-1}(\mathbf{u}_j) \le \mathbf{X} \le \underline{F}_X^{-1}(\mathbf{u}_j)\}, \\ \overline{g}_j(\mathbf{u}_j) &= \max\{g(\mathbf{X}) : \overline{F}_X^{-1}(\mathbf{u}_j) \le \mathbf{X} \le \underline{F}_X^{-1}(\mathbf{u}_j)\}. \end{aligned} \tag{22}$$

The interval quasi-MC sampling procedure can be summarized as follows.

Step 1: Generate a low-discrepancy point set $P_n = \{\mathbf{u}_j, j = 1, \ldots, n\}$.

Step 2: Generate interval samples $[\overline{F}_X^{-1}(\mathbf{u}_j),\ \underline{F}_X^{-1}(\mathbf{u}_j)]$.



a) Random samples (n = 100) b) GLP (n = 89)

c) Halton sequence (n = 100) d) Faure sequence (n = 100)

Figure 1. 2D scatter plots of different point sets.

Step 3: For the jth simulation, compute the extrema of $g(\mathbf{X})$ for $\mathbf{X} \in [\overline{F}_X^{-1}(\mathbf{u}_j),\ \underline{F}_X^{-1}(\mathbf{u}_j)]$.

Step 4: Compute the lower and upper bounds for P_f by Eq. (21).

The proposed method requires the additional computational cost for constructing low-discrepancy sequences, as compared to the direct interval MC method. However, in a practical structural reliability analysis, the CPU time needed for the construction of low-discrepancy sequences is generally negligible in comparison with that of performing multiple simulations (Dai and Wang, 2009). Open-source libraries/routines for computer implementation of popular low-discrepancy point sets are available in programming environments, such as C++, Fortan, Matlab and others (Burkardt, 2011).

Step 3 in the proposed method involves the calculation of the range of the limit state function g when the inputs vary in certain closed intervals. The problem of finding the range of values of a function is solved on the basis of interval analysis (Moore, 1966). In many practical cases of structural reliability assessment, the limit state of interest (e.g., stress, displacement) is computed through finite element (FE) analysis, thus the limit state function is only implicitly known. In this case, an interval FE analysis is needed for the purpose of evaluating Eq. (9). An implementation of these techniques in the proposed method is straightforward. Theory and algorithms for linear elastic interval FE analysis are well developed. Reliable methods are available to compute the bounds of responses of realistic structures with reasonable accuracy when the structural stiffness and geometrical properties and loads vary in relatively narrow intervals. Reviews of interval FE analysis can be found in (Zhang, 2005) and (Moens and Vandepitte, 2005). It should be noted that, even

with the relatively efficient interval FE analysis, the computing cost is generally considerably higher than the corresponding deterministic FE analysis. It is crucial to limit the number of simulations needed.

At present, the proposed interval quasi-MC method can only handle independent p-boxes, but extensions to handle problems with dependencies are under development. Dependence information between p-boxes can be implemented via copulas as discussed in Ferson et al. (2004). Another option for consideration of dependencies in the p-box framework is the construction of multivariate models for associated random sets (Fetz and Oberguggenberger, 2010). Or one may consider the p-boxes as generated by a parametrized set of real-valued random variables and implement the dependencies between them in the traditional manner. This is associated with the concept proposed in (Möller et al., 2004) for fuzzy probabilities. In general, these concepts of imprecise probabilities provide even more flexibility in specifying dependencies compared to traditional probabilistic concepts. They can handle vaguely defined, i.e. imprecise, dependencies, such as a fuzzy correlation (Möller et al., 2006), and even cases in which only marginals are known but the copula unknown (Klir, 2006). In those cases, the indeterminacy, or imprecision, in the dependencies are translated into imprecision in CDF's, i.e. into p-boxes.

4.3. VARIANCE FOR INTERVAL QUASI-MONTE CARLO

Because standard low-discrepancy sequences are deterministic, a variance-type error estimate cannot be calculated directly for the interval quasi-MC using Eq. (13). This is one of the limitations of using low-discrepancy sequences. To address this issue, we randomize the low-discrepancy sequences to create a random sample of low-discrepancy sequences, each will be used to compute a reliability bound. Empirical variances of the obtained results can then be calculated to measure the quality of the interval quasi-MC.

A simple randomization method is to use a random shift (Cranley and Patterson, 1976). Consider an s-dimensional low-discrepancy point set $P_n = \{\mathbf{u}_j, j = 1, \ldots, n\}$. Let $\mathbf{v}$ be an s-dimensional uniform random vector. The randomized points $\tilde{\mathbf{u}}_j$ can be constructed as

$$\tilde{\mathbf{u}}_j = (\mathbf{u}_j + \mathbf{v}) \bmod 1, \quad \text{for } j = 1, \ldots, n \tag{23}$$

in which the modulo 1 operation is taken componentwise. Other more advanced randomization methods include random scrambling and permutations, see (Lemieux, 2009).

Using the random shift method, we can create m independent randomized copies of P_n, i.e., $\{\tilde{\mathbf{u}}_{j,1}, j = 1, \ldots, n\}, \ldots, \{\tilde{\mathbf{u}}_{j,m}, j = 1, \ldots, n\}$. For the lth copy of P_n, a lower and an upper bound of P_f are calculated as:

$$\begin{aligned} \underline{P}_{f,\,l} &= \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}[\overline{g}_j(\tilde{\mathbf{u}}_{j,\,l}) \le 0], \\ \overline{P}_{f,\,l} &= \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}[\underline{g}_j(\tilde{\mathbf{u}}_{j,\,l}) \le 0]. \end{aligned} \tag{24}$$

The variance of $\underline{P}_{f,\,l}$ can be estimated by its empirical variance

$$\widehat{\mathrm{Var}}(\underline{P}_{f,\,l}) = \frac{1}{m-1}\sum_{l=1}^{m}(\underline{P}_{f,\,l} - \underline{P}_f)^2, \tag{25}$$

where

$$\underline{P}_f = \frac{1}{m}\sum_{l=1}^{m} \underline{P}_{f,\,l}. \tag{26}$$

The empirical variance of the upper bound $\underline{P}_{f,\,l}$ can be calculated in a similar manner.

5. Examples

5.1. RIGID-PLASTIC PORTAL FRAME

A rigid-plastic portal frame, shown in Fig. 2, is subjected to a horizontal load H and a vertical load V. The example is adopted from (Melchers, 1999). The failure probability of the structure is mainly controlled by three limit states defined as follows:

$$\begin{aligned} g_1(\mathbf{X}) &= M_1 + 2M_3 + 2M_4 - H - V, \\ g_2(\mathbf{X}) &= M_2 + 2M_3 + M_4 - V, \\ g_3(\mathbf{X}) &= M_1 + M_2 + M_4 - H, \end{aligned}$$

in which M_1,...,M_4 are the plastic bending capacities at the joints. Since this is a series system, the system limit state function g is the minimum of the above, i.e.,$g = \min(g_1, g_2, g_3)$. The basic random variables are $\mathbf{X} = (M_1, \ldots, M_4, H, V)$. All the variables are normally distributed and mutually independent. Assume that because of limited knowledge, the mean values of X_i can only be estimated as intervals. Table I summarizes the statistics for the basic random variables.

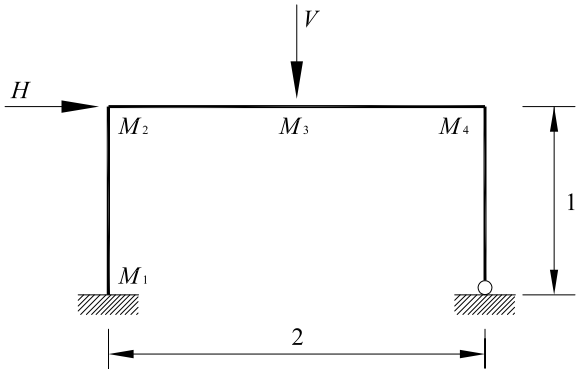


Figure 2. A rigid-plastic portal frame.

The proposed quasi-interval Monte Carlo method is applied using the GLP, Halton and Faure points with different numbers of samples. The random shift method (Eq. 23) was used to randomize the sample points. The first 10,000 points in the Halton and Faure sequences were omitted, as suggested in (Shinoda, 2007). The Halton points were also leaped with $L = 61$. As an independent check, direct interval Monte Carlo simulation was conducted. Table II compares the computed reliability bounds from the two methods with

Table I. Random variables for the rigid frame.

Variable	Mean	Standard dev.
M_i	[0.98,1.02]	0.15
H	[1.87,1.93]	0.45
V	[1.18,1.22]	0.3

different numbers of simulations. With the direct interval Monte Carlo, the failure probability is found to be [0.0118, 0.0287] after 10^5 simulations. It can be seen from Table II that the interval quasi-Monte Carlo methods with GLP, Halton and Faure sequences all give reasonable results after 3,000 simulations. The results were somewhat improved when the number of simulations was increased to 10,000.

To achieve a variance-type error estimate, each method was then repeated 30 times to obtain a sample of 30 results. On the basis of the 30 calculations, the sample variance for the estimated reliability bounds by each method were calculated and compared in the last two columns of Table II. For the direct interval Monte Carlo results, the estimates of variance calculated using Eqs. (13) and (14) are also presented to compare with the empirical results. It is evident that with the same number of simulations, the (randomized) interval quasi-Monte Carlo results have less variance than the direct interval Monte Carlo results. For instance, with 3000 simulations, the sample variance for $\underline{P}_f$ is 0.00224 for direct interval Monte Carlo; it reduces to around 0.001 for the present method. From Table II, it can also be seen that for the direct interval Monte Carlo, variance estimates obtained from Eqs. (13) and (14) agree well with the empirical results.

Table II. Reliability bounds and their empirical variances (Example 1: rigid frame).

Method-No Simul.	$\underline{P}_f \times 10^2$	$\overline{P}_f \times 10^2$	$\widehat{\mathrm{Var}}_{\underline{P}_f} \times 10^2$	$\widehat{\mathrm{Var}}_{\overline{P}_f} \times 10^2$
GLP-3001	1.13	2.87	0.105	0.140
GLP-5003	1.12	2.96	0.895	0.124
GLP-10007	1.19	2.86	0.051	0.045
Halton-3000	1.20	2.70	0.109	0.169
Halton-5000	1.18	3.06	0.083	0.132
Halton-10^4	1.19	2.94	0.043	0.075
Faure-3000	1.27	2.57	0.118	0.175
Faure-5000	1.28	2.78	0.104	0.161
Faure-10^4	1.21	2.83	0.076	0.123
interval MC-3000	1.33	3.03	0.224 (0.195) *	0.302 (0.302)
interval MC-10^4	1.11	2.84	0.112 (0.109)	0.182 (0.167)
interval MC-10^5	1.18	2.87	0.038 (0.034)	0.061 (0.053)

* values in parentheses are based on Eqs. (13) and (14).

5.2. TEN-BAR TRUSS UNDER WIND LOAD

The second example is a planar 10-bar truss subject to wind load, as shown in Fig. 3. The structure layout and member sizes are adopted from (Nie and Ellingwood, 2005). The response of the truss is assumed to be linear elastic. The limit state in question is the roof drift with a limit value of 17.78 mm. We are interested in finding the failure probability under the (annual maximum) wind load. The member cross-sectional areas and the wind load are taken to be random variables. Let A_i denote the cross-sectional area for the ith member. The basic random variables are $\mathbf{X} = (A_1, \ldots, A_{10}, W)$. Table III summarizes the statistics for the cross-sectional areas. All cross-sectional areas are assumed to be mutually statistically independent.

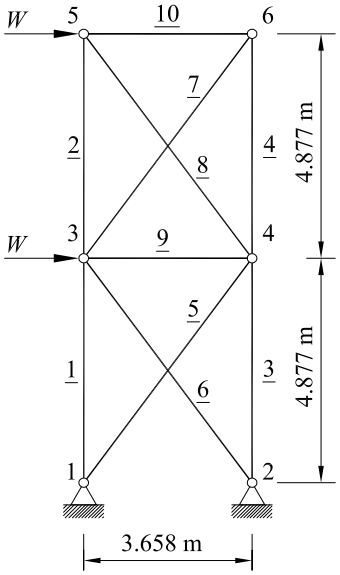


Figure 3. Ten-bar plane truss subject to wind load.

Table III. Statistics for the cross-sectional areas for the 10-bar truss (unit: cm^2).

Variable	Mean	COV	Distribution
$A_1 - A_4$	53.226	0.1	Normal
$A_5 - A_8$	45.677	0.1	Normal
A_9, A_{10}	9.548	0.1	Normal

Suppose that a 30-year record of the annual maximum wind speed at the location of the structure is available, which represents a sample of size 30 for the load, see Table IV. The sample mean and sample standard deviation for the wind load W are 112.99 kN and 35.51 kN, respectively.

Based on this rather restricted set of wind load data, the probability boxes for the wind load were constructed using the four approaches introduced in Section 2. Fig. 4 shows the p-boxes derived by the

Table IV. Samples of annual maximum wind load, W (kN).

Year	W	Year	W	Year	W
1	81.33	11	128.79	21	149.51
2	121.94	12	97.37	22	80.05
3	175.01	13	72.13	23	134.91
4	91.50	14	88.03	24	74.02
5	101.02	15	110.05	25	150.12
6	110.66	16	130.78	26	100.81
7	106.58	17	114.06	27	86.55
8	83.71	18	69.09	28	198.92
9	94.88	19	123.18	29	80.61
10	151.61	20	198.16	30	84.17

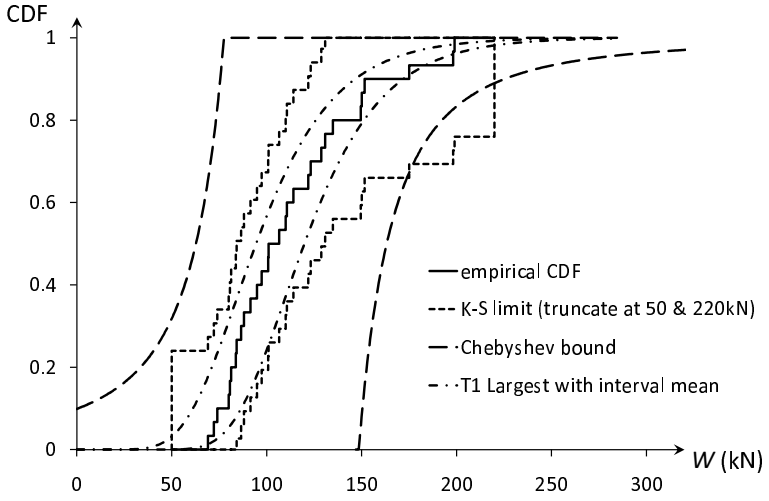


Figure 4. CDF bounds for the wind load W constructed using different methods.

Kolmogorov-Smirnov confidence limits (at 5% significance level) and Chebyshev's inequality. Since the K-S confidence limits only give the p-box for the range of the samples, it is necessary to assume a smallest and a largest possible value to truncate the K-S limits. In this example, the wind load samples have a range of 69.09 to 198.92 kN. The K-S limits are truncated at 50 and 220 kN. These two limit values were chosen based on the authors' engineering judgement.

Next, the confidence interval on the mean value was used to define the p-box. Assuming the population standard deviation σ_W is equal to the sample standard deviation, the 95% confidence interval on the population mean is approximately [100.28, 125.69] kN. Assume that based on experience, the (annual maximum)

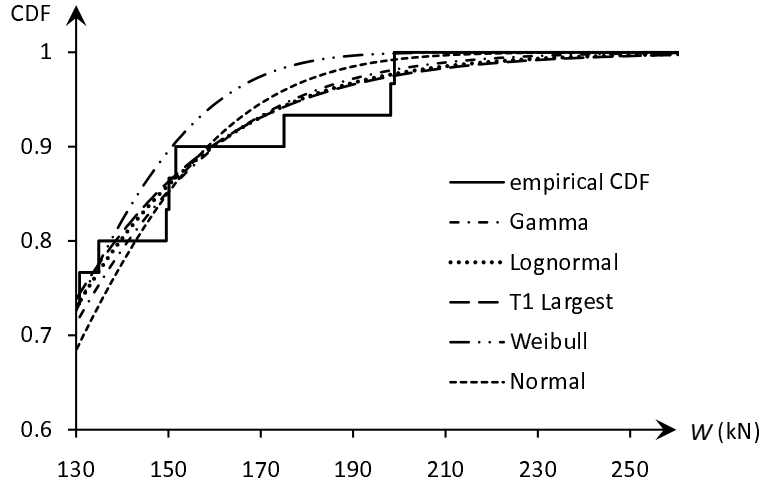


Figure 5. Candidate distributions for the wind load (upper tails).

wind load can be properly modeled by a Type 1 Largest (T1 Largest) extreme value distribution. The p-box for the T1 Largest with an interval mean [100.28, 125.69] kN and a standard deviation of 35.51 kN is illustrated in Fig. 4.

The wind load p-box can also be constructed using the envelope of a set of candidate distributions. Five candidate distributions were chosen to fit the wind load data, i.e., the T1 Largest, lognormal, Gamma, Normal and Weibull distributions. The Kolmogorov-Smirnov goodness-of-fit tests were performed to test the validity of these assumed distributions. It can be shown that all candidate distributions passed the K-S goodness-of-fit tests at the significance level of 5%. As the K-S statistics for the different distributions are relatively similar, their CDFs would hardly be distinguished if plotted in one figure. However, the difference between the five candidate distributions in their upper tails can be clearly seen from Fig. 5. As the structural reliability is dominated by the upper tail of the distribution of the wind load, it is expected that the failure probabilities obtained from different wind models will differ significantly.

The proposed interval quasi-Monte Carlo method is applied with the four different p-box models for the wind load. The results by GLP, Halton and Faure sequences are very similar. Only the latter is presented herein. The first 10,000 points in the Faure sequence were omitted. The next 5,000 points were then used for the simulation. The results were verified by direct interval Monte Carlo simulation with 10^5 samples. The two sets of the results are compared in Table V. It may be observed from Table V that the results from the present method agree well with those from direct interval Monte Carlo.

It is more interesting to note that the bounds of P_f for different wind load p-box models vary considerably. Using Chebyshev's inequality yields the widest P_f, i.e., $P_f = [0,\ 0.37]$. It is unlikely that the analyst will find such a wide P_f practically useful. This is because the wind load p-box based on Chebyshev's inequality is very wide, particularly in the upper tail as can be seen in Fig. 4. It is due to the fact that only the information of the first two moments about the random variable is used. Note that the p-box derived by Chebyshev' inequality (Eq. 3) is independent of the sample size given that the first two moments are known. Two sets of data, one with limited samples and a second with comprehensive samples, would lead to the

same p-box if the first two moments from the two sets of data were the same. This lack of distinction is undesirable. Moreover, with Chebyshev's inequality, the p-box will not necessarily become narrower when additional data are available. This does not agree with the general conception that epistemic uncertainty can be reduced when the quality of data is refined. Because of these limitations, it appears that Chebyshev's inequality is not suitable for the construction of p-box in structural reliability assessment.

With the wind load p-box constructed using the K-S confidence limits, P_f is found to vary in a wide range between 0 and 0.246. Unlike Chebyshev's inequality, the K-S confidence limits will become narrower as the number of samples increase. However, the K-S approach has a main drawback, i.e., the derived p-boxes have to be truncated at the tails, and the result may be influenced strongly by the values of the truncation, which are often chosen arbitrarily. In this example the K-S limits are truncated at 220 kN at the upper end, and $\overline{P}_f$ is found to be 0.246. If the tail is truncated at 198.92 kN, i.e. the maximum of the sampled wind loads, $\overline{P}_f$ will reduce from 0.246 to 0.0288. Similar observation is made in (Ferson et al., 2003). Because of the sensitivity of results to the truncation, it is not practical to use the Kolmogorov-Smirnov confidence limit to construct p-boxes for the purpose of reliability analysis.

Table V. Reliability bounds of the 10-bar truss.

P-box construction	quasi-interval MC (Faure, $n = 5000$)	interval MC ($n = 10^5$)
K-S limits	[0, 0.246]	[0, 0.246]
Chebyshev	[0, 0.368]	[0, 0.359]
P-box 3	$[1.16, 2.66] \times 10^{-2}$	$[1.1, 2.67] \times 10^{-2}$
P-box 4	$[0.06, 1.62] \times 10^{-2}$	$[0.052, 1.73] \times 10^{-2}$

P-box 3: T1 Largest with interval mean.

P-box 4: envelope of 5 candidate distributions.

If the effect of small sample size is considered through the confidence interval of the first-order statistics, the wind load is modeled by a Type 1 Largest with an interval mean of [100.28, 125.69] kN (95% confidence interval). Under this assumption, P_f varies between 0.0116 and 0.0266. This interval bound clearly demonstrates the effect of small sample size on the calculated structural reliability. It indicates that confidence intervals on distribution parameters is a reasonable way to define p-boxes, provided that the appropriate distribution form can be discerned.

In the case where the wind load p-box is constructed using the envelope of five candidate distributions, the variation of P_f is from 6×10^{-4} to 0.0162. The very small lower bound of P_f is contributed by the Weibull distribution. As evidenced from Fig. 5, the Weibull distribution is the least conservative one in the upper tail region. If the Weibull is discarded, the bounds of P_f will become [0.32, 1.62] $\times 10^{-2}$. These results highlight the sensitivity of the failure probability to the choice of the probabilistic model for the wind load. Although the candidate distributions all passed the K-S goodness-of-fit test and their K-S statistics are relatively similar, their tails can be different, leading to very different reliability results.

6. Conclusions

A new interval quasi-Monte Carlo method has been developed to perform structural reliability analysis with probability boxes. The methodology is based on deterministic low-discrepancy sequences, which are more uniformly distributed than pure random samples. Compared with the direct interval Monte Carlo method, the present method can effectively reduce the required number of simulations to achieve comparable accuracy. The proposed interval quasi-Monte Carlo method can also be randomized so that variance-type error estimates can be obtained. The implementation of the present method is relatively straightforward; it replaces the pseudo random numbers in interval Monte Carlo by the low-discrepancy sequences. It is a general technique that can be applied to a variety of uncertainty models, including probability boxes, random sets, random intervals, Dempster-Shafer evidence theory, fuzzy random variables, etc.

A frame and a truss have been analyzed to demonstrate the efficiency and accuracy of the proposed method. For both problems, the present method yields reasonably good results with at least one order-of-magnitude less computational effort than the direct interval Monte Carlo simulation.

For the truss example, the wind load p-box was constructed using four different approaches suggested in the literature, i.e., the Kolmogorov-Smirnov confidence limits, Chebyshev's inequality, confidence interval on the mean value, and the envelope of five candidate distributions which all pass K-S goodness-of-fit tests. The reliability results suggest that the K-S approach or Chebyshev's inequality do not lend themselves to the practical construction of probability boxes in the context of structural reliability analysis. The p-boxes based on these two approaches seem to be too conservative. It is unlikely that the analysts/decision-makers will find such conservative P_f practically useful. There are also practical difficulties in using the K-S confidence limits and Chebyshev's inequality to define p-boxes. The most reasonable method to construct p-boxes for the purpose of reliability assessment seems to be their construction based on confidence intervals of statistics.

Appendix. Low-discrepancy sequences

This section briefly introduces the constructions of the GLP, Halton, and Faure sequences. See (Niederreiter, 1992) and (Lemieux, 2009) for details.

GOOD LATTICE POINT

Let n be an integer ≥ 2, $\mathbf{a} = (a_1, \ldots, a_s)$ be an integer vector, and gcd(a_i, n)=1 for all i, where gcd = greatest common divisor. A point set $P_n = \{\mathbf{u}_j, j = 1, \ldots, n\}$ can be constructed by

$$\mathbf{u}_j = \mathrm{frac}(\frac{j-1}{n}\mathbf{a}) \quad \text{for } j = 1, \ldots, n \tag{27}$$

where $\mathrm{frac}(z)$ denotes the fractional part of z. Such a point set is known as a lattice point set with the generating vector **a**. Among all possible lattice point sets, the one that has the smallest discrepancy is called the GLP point set. It has been suggested that for a prime number n, the generating vector can be taken as (Korobov, 1959):

$$\mathbf{a} = (1, a, \ldots, a^{s-1}) \bmod n, \tag{28}$$

where a is an integer, $1 \leq a \leq n-1$, and $\gcd(a,n) = 1$. It was shown that an optimal value of a exists for given values of s and n such that the resulting good lattice point set has a discrepancy $D_n^{(s)} = O(n^{-1}(\log(n))^s)$. The method for determining the optimal generating vectors can be found in (Fang and Wang, 1994).

HALTON SEQUENCE

Let $b \geq 2$ be an integer, then any integer $n \geq 0$ has a unique digit expansion in base b as

$$n = \sum_{j=0}^{r} a_j(n) b^j = a_0(n) + a_1(n)b + \ldots + a_r(n)b^r, \tag{29}$$

where $a_j(n) \in \{0, 1, \ldots, b-1\}$ for all i, and $b^r \leq n < b^{r+1}$. Thus, r can be calculated as

$$r = \lfloor \log_b n \rfloor, \tag{30}$$

in which $\lfloor x \rfloor$ denotes the integral part of x. Define the radical-inverse function $\phi_b(n)$ in base b as

$$\begin{aligned} \phi_b(n) &= \sum_{j=0}^{r} a_j(n) b^{-j-1} \\ &= a_0(n)b^{-1} + a_1(n)b^{-2} + \ldots + a_r(n)b^{-r-1}. \end{aligned} \tag{31}$$

Note that for all integers $n \geq 0$, $\phi_b(n) \in [0, 1)$. With this definition of radical-inverse function, the nth vector of the s-dimensional Halton sequence is defined as

$$\mathbf{u}_n = (\phi_{b_1}(n), \phi_{b_2}(n), \ldots, \phi_{b_s}(n)), \; n = 1, 2, \ldots \tag{32}$$

with the bases $b_1, \ldots, b_s > 1$ being pairwise relatively prime. In practice, the first s primes are usually used as the bases. It has been shown that the Halton sequence formed by the first n points ($n > \max(b_1, \ldots, b_s)$) has the discrepancy $O(n^{-1}(\log(n))^s)$ (Halton, 1960).

FAURE SEQUENCE

The construction of Faure sequence appears to be like the Halton sequence, but it uses only one base for all dimensions. Given a dimension s, let b be a prime number such that $b \geq s$ and $b \geq 2$. The first dimension of the Faure sequence corresponding to n is given by

$$\phi_b^1(n) = \sum_{j=0}^{r} a_j^1(n) b^{-j-1}, \quad n = 1, 2, \ldots \tag{33}$$

in which the superscript 1 denotes the first dimension, and $a_j^1(n)$ equals to the expansion coefficient defined in Eq. (29), i.e.,

$$a_j^1(n) = a_j(n). \tag{34}$$

The elements of the higher dimensions of the Faure sequence are then constructed as

$$\phi_b^k(n) = \sum_{j=0}^{r} a_j^k(n) b^{-j-1}, \quad 2 \le k \le s,\ n = 1, 2, \ldots \tag{35}$$

and the coefficients $a_j^k(n)$ are calculated recursively from those of the lower dimensions:

$$a_j^k(n) = \sum_{i \ge j}^{r} C(i,j) a_i^{k-1}(n) \bmod b \tag{36}$$

in which $C(i,j) = i!/j!(i-j)!$ is the combinatorial function. Thus the nth vector of the sample points in the sequence is

$$\mathbf{u}_n = (\phi_b^1(n), \phi_b^2(n), \ldots, \phi_b^s(n)),\ n = 1, 2, \ldots \tag{37}$$

With digital sequences such as Halton and Faure sequences, it has been suggested to omit some initial sequence points as they often exhibit undesirable correlations among different dimensions. Another technique is to use "leaped" sequence points, i.e., to use only every Lth points in the sequence, where L is the leap (Kocis and Whiten, 1997).

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