Bounding the dependence measures for spatial uncertainties

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Abstract. The analysis and design of mechanical engineering systems requires to take into account the influence of uncertainties on the system's performance. Depending on the available amount of information, the designer or analyst can choose from a wide variety of methods in the probabilistic (see e.g. (Schuëller, 2001)) or non-probabilistic (see e.g. (Elishakoff and Ohsaki, 2010), (Moens and Hanss, 2011)) approaches to describe the uncertainties. However, the selection of a suitable uncertainty model for the different uncertainties most often is not enough to yield satisfactory information on the reliability or bounds of the system's performance. A crucial piece of information appears to be the dependence of the uncertain variables. This is especially the case for uncertainties with a spatial character, e.g. material properties or distributed loads.

The study gives an overview of the existing probabilistic and non-probabilistic methods to represent this kind of dependencies. In the probabilistic setting the concepts of the covariance function associated with a random field, a copula and several correlation measures are treated. In the non-probabilistic setting the concepts of interval fields, convex modeling and interactive fuzzy numbers are reviewed. Of special interest is the ability to bound these dependence measures. For the case of a spatial uncertainty, this generally comes down to specifying the maximum distance between points that are influencing each other. Points further away from each other than this distance are considered practically independent. For points closer to each other than this distance the interaction may be described, introducing a notion of perfect dependence. Finally, of utmost importance is to study the effect of the bounds on the dependence on the uncertainty in the system's performance.

Keywords: Random field, Interval field, Finite Element analysis

1. Introduction

A description of dependence can take many faces. First of all, the word itself has different meanings. Mosteller and Tukey (1977) emphasize: "We must be clearer about the abused word *dependence* and its relatives." (Drouet Mari and Kotz, 2001) On the one hand, dependence may mean perfect dependence, i.e. if one knows the value of x, then one knows exactly the value of y. On the other hand, dependence can be more flexible, i.e. when we know x, we may know something more about y as opposed to the situation

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when we know nothing about the value of x. This piece of information can be expressed in a probabilistic framework (section 2) as well as a non-probabilistic framework (section 3).

The modeling of a spatial uncertainty exemplifies the need for a proper description of dependence in uncertainty modeling. It is a given that spatially distributed model parameters for locations adjacent to each other assume uncertain but very similar values. The uncertain values for points further away from each other can be very dissimilar. The crucial piece of information is the notion of spatial closeness. This notion consists in actually two things. The first is being able to measure how close points are to each other in a numerical model. The second is to be able to compare this distance to a reference to assess whether higher or lower dependence is assumed between these two points. For the details of a distance measuring method in a finite element model, the reader may consult (De Mulder et al., 2012). Specifying a value for the reference distance depends on the actual uncertainty modeling framework in use. In a probabilistic framework, the concept of correlation length is widely used, whereas in the non-probabilistic world a related concept does not (yet) exist. Next issue is to take into account this dependence when propagating input uncertainty to output uncertainty: According to (Kurowicka and Cooke, 2006) "...an essential part of uncertainty analysis is the analysis of dependence. Indeed, if all uncertainties are independent, then their propagation is mathematically trivial (though perhaps computationally challenging). Sampling and propagating independent uncertainties can easily be trusted to the modellers temselves. However, when uncertainties are dependent, things become much more subtle, and we enter a domain for which the modellers' training has not prepared them." From a practical perspective, a tool is needed to translate the spatial dependence information given by an expert to a representative set of realisations of the uncertain model parameter. Section 4 sheds some light on this topic.

An important feature of the spatial dependence modeling is yet untouched. Although the more flexible notion of dependence (i.e. *not* the perfect dependence: if $x = a$ then $y = b$) and its related probabilistic and non-probabilistic descriptions allows a more versatile treatment of the dependence phenomenon, it becomes increasingly clear that the assignment of one single value to a dependence descriptor is still far from feasible. Either because the data set on which such a single value assignment could be made is too small or the data set simply does not exist and the dependence information is based on expert knowledge. To quote again (Kurowicka and Cooke, 2006) "Engineers and scientists sometimes cover their modesty with churlish acronyms designating the source of ungrounded assessments. 'Wags' (wild-ass guesses) and 'bogsats' (bunch of guys sitting around a table) are two examples found in published documentation." It is suggested in section 5 to put bounds on the dependence measures, instead of assigning a single value to them.

Finally, in section 6 a numerical example is given to show the effect of such bounds on a probabilistic and non-probabilistic dependence measure in the context of spatial uncertainty modeling.

2. Probabilistic dependence measures

In scientific literature notions and definitions of *independence* preceded the notion of *dependence* (the latter was just regarded as the negation of the former). Related to this seems the fact that it was easier to understand independence. "Saying that variables are *not* independent does not say much about their joint distribution. What is the nature of this dependence? How dependent are they? How can we measure the dependence? These questions must be addressed in building a dependence model." (Kurowicka and Cooke, 2006) The first probabilistic concept of dependence (correlation) emerged at the end of the 19^{th} century in the field of social study and psychology. “The concept of correlation (and its modifications) introduced by F. Galton in 1885 dominated statistics during some 70 years of the 20-th century, practically serving as the only measure of dependence, often resulting in somewhat misleading conclusions.” (Drouet Mari and Kotz, 2001) On the other hand, the correlation has great merit. “By now, over a century later, contemporary scientists often take the correlation coefficient for granted. It is not appreciated that before Galton and Pearson, the only means to establish a relationship between variables was to deduce a causative connection. There was no way to discuss - let alone measure - the association between variables that lack a cause-effect relationship.” (Rodgers and Nicewander, 1988) In the following, an overview of several dependency measures is given. First, the scalar aggregate (global) measures of bivariate dependence *product moment correlation*, *rank correlation*, *Kendall's tau* and *relative entropy* are discussed. Next, the more thorough (local) measure of dependence *copula* is discussed.

2.1. Product moment correlation

A historic account of the product moment correlation, together with as much as 13 ways to look at it can be found in (Rodgers and Nicewander, 1988). The product moment correlation of random variables X,Y with finite expectations $E(X)$, $E(Y)$ and finite variances σ_X^2, σ_Y^2, is

$$\rho(X,Y) \equiv \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} \tag{1}$$

Some properties of the product moment correlation are listed below:

- The product moment correlation depends on the marginal distribution functions F_X and F_Y.
- The product moment correlation is bounded by $-1 \leq \rho(X,Y) \leq 1$ and it's minimum and maximum (not necessarily -1 and 1, as in the example (Kurowicka and Cooke, 2006), pages 29-30) are attained for X and Y countermonotonic and comonotonic, respectively. We say that random variables X and Y are comonotonic if there is a strictly increasing function G such that $X = G(Y)$. X and Y are countermonotonic if X and $-Y$ are comonotonic.
- The product moment correlation is invariant under linear strictly increasing transformations of X or Y, but is not invariant under non-linear strictly increasing transformations.
- If X and Y are independent, then $\rho(X,Y) = 0$. The reverse is not generally true.

2.2. Rank correlation

The rank correlation or Spearman correlation was introduced by Spearman in 1904. The rank correlation of random variables X, Y with cumulative distribution functions F_X and F_Y is

$$\rho_r(X,Y) \equiv \rho(F_X(X), F_Y(Y)) \tag{2}$$

Since for any X with a continuous invertible F_X, $F_X(X)$ is uniform on $[0,1]$, the rank correlation is a correlation of random variables transformed to uniform random variables. This leads to the following properties:

- The rank correlation is independent of marginal distributions.
- The rank correlation is bounded by $-1 \leq \rho_r(X, Y) \leq 1$ and it's minimum (-1) and maximum (1) are attained for X and Y countermonotonic and comonotonic, respectively.
- The rank correlation is invariant under non-linear strictly increasing transformations.
- If X and Y are independent, then $\rho_r(X, Y) = 0$. The reverse is not generally true.

In (Kurowicka and Cooke, 2006) an efficient technique is presented to extract information on the rank correlation from experts. The technique is based on an indirect question: "Suppose Y were observed in a given case and its values were found to lie above the median value for Y; what is your probability that, in this same case, X would also lie above its median value?" Formally this comes down to assess

$$\pi_{\frac{1}{2}}(X, Y) \equiv P\left(F_X(X) > \frac{1}{2} | F_Y(Y) > \frac{1}{2}\right) \tag{3}$$

Based on the minimum information copula (see below), the probability assigned by the expert can be directly related to a rank correlation. It is obvious that a probability 0 would mean X and Y are anti-correlated, a value $\frac{1}{2}$ would suggest a rank correlation equal to 0 whereas a value 1 leads to completely rank-correlated X and Y.

2.3. Kendall's tau

Let (X_1, Y_1) and (X_2, Y_2) be two independent pairs of random variables with joint distribution function F and marginal distributions F_X and F_Y. Kendall's rank correlation, also called Kendall's tau (1938) is given by

$$\tau \equiv P\left[(X_1 - X_2)(Y_1 - Y_2) \geq 0\right] - P\left[(X_1 - X_2)(Y_1 - Y_2) < 0\right] \tag{4}$$

The following properties hold for Kendall's tau:

- Kendall's tau is independent of marginal distributions.
- Kendall's tau assumes values between -1 and 1.
- Kendall's tau is invariant under non-linear strictly increasing transformations.
- If X and Y are independent, then $\tau(X, Y) = 0$. The reverse is not generally true.

For a discussion of two correlation measures, namely *sup correlation* and *monotone correlation*, where a zero value implies independence, see (Devroye, 1986) pp.574-576.

2.4. Relative entropy

Consider the pair of random variables (X, Y) with $f(x, y)$ the joint density and $f_1(x)$ and $f_2(x)$ the marginal densities. Then the relative entropy is defined as:

$$\delta_{X,Y} \equiv \int\int f(x, y)\, log\left(\frac{f(x, y)}{f_1(x)f_2(y)}\right) dxdy \tag{5}$$

The entropy of the density $f(x, y)$ is compared with the maximum attainable entropy, namely when X and Y are independent. For independent X and Y, $\delta_{X,Y}$ is zero, and for maximal dependence, $\delta_{X,Y}$ approaches infinity. The concept of maximum entropy or equivalently minimum information will be reappearing in the next section on copulae.

2.5. COPULA

Copulae are tools for modeling dependence of several random variables. In particular a copula allows to seperate the effect of dependence from the effect of marginal distributions in a joint distribution. The term copula was first introduced by Sklar in 1959 (Schmidt, 2006). A copula C is defined as a function which is a cumulative distribution function with uniform marginals. Random variables X and Y are joined by copula C if their joint distribution can be written as

$$F_{XY}(x, y) = C(F_X(x), F_Y(y)) \tag{6}$$

For the bivariate case, a copula is the joint distribution of two random variables that are each converted to the uniform distribution by means of their respective marginal distribution functions. A copula is always contained in between the Fréchet-Hoeffding bounds C_L and C_U (see Figure 1). C_L represents the case when all of the probability mass is spread uniformily on the main diagonal $v = 1 - u$ and C_U is attained when all of the mass is on the main diagonal $v = u$.

$$C_L = max(u + v - 1, 0) \leq C(u, v) \leq C_U = min(u, v) \tag{7}$$

with $(u, v) \in \lfloor 0, 1\rceil^2$. Next, two copulae from a sheer endless list of copulae are described (based on (Kurow-

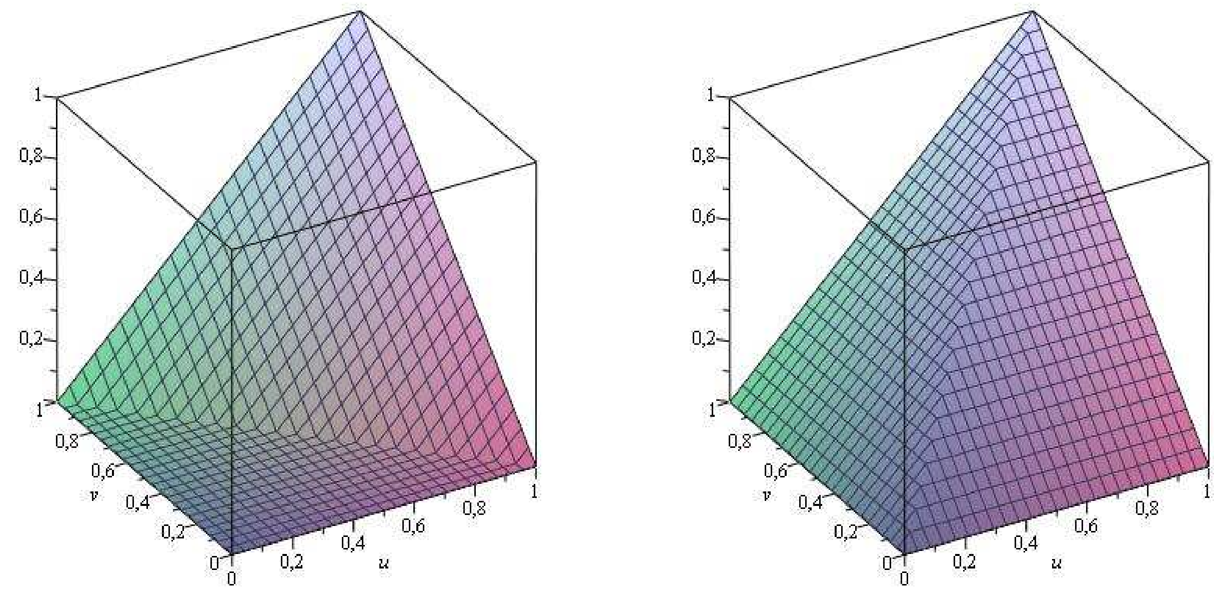

Figure 1. The lower and upper Frchet-Hoeffding bounds C_L and C_U

icka and Cooke, 2006)) as they are of particular interest in next section's discussion on non-probabilistic dependence measures.

2.5.1. *Diagonal band copula*
One natural generalization of the bounding copula C_U is the diagonal band copula introduced by Cooke and Waij in 1986. In contrast to C_U, for possitive correlation the mass is concentrated on the diagonal band with vertical bandwidth $b = 1 - a$. Mass is distributed uniformly on the inscribed rectangle and is uniform but 'twice as thick' in the triangular corners. A clear relationship exists between the product moment correlation and the parameter a of the diagonal band copula.

2.5.2. *Minimum information copula*
The minimum information copula was introduced and studied by Meeuwissen in 1993. The construction is based on the fact that for any $\rho_r \in \lfloor -1, 1 \rceil$ there is a unique bivariate joint distribution satisfying the following constraints:

– the marginal distributions are uniform on $I = [-\frac{1}{2}, \frac{1}{2}]$
– the rank correlation is $\rho_r \in \lfloor -1, 1 \rceil$
– the distribution has minimal information relative to uniform distribution (or maximum entropy as defined higher) among all distributions with rank correlation ρ_r.

The minimum information copula is attractive because it realizes a specified rank correlation by 'adding as little information as possible' to the product of the marginals. Its main disadvantage is that it does not have a closed functional form. All calculations with this copula must involve numerical approximations.

3. Non-Probabilistic dependence measures

All too often the following typical 'jump of thought' is made, it is first noted that "…, the dependence is obviously not deterministic but of a stochastic nature." (Drouet Mari and Kotz, 2001), and then a book all about probabilistic dependence follows. Present authors maintain however that if something is not deterministic it is not necessarily probabilisitic. The non-probabilistic methods cover a very large set of alternative uncertainty treatments, of which (together with probabilistic theory) an excellent overview is given in (Klir, 2006). We will limit our focus to crisp and so called graded possibilities. In crisp possibilities a clear distinction is made between members and non-members of a set, by assigning membership level one and zero, respectively. A typical example of a crisp set is an interval. In graded possibilities the full range of membership degrees between zero and one is available. A typical example of a graded set is a fuzzy number.

In this framework, the treatment of independence and noninteraction overshadows, to the best of our knowledge, study in the field of dependence descriptions and measures. We follow mainly the description by (Klir, 2006) on both crisp possibilities and graded possibilities.

3.1. Crisp possibilities and information transmission

For finite sets (generalization to infinite sets is given in (Klir, 2006)) of possible alternatives Hartley proposed in 1928 to measure the amount of uncertainty of such a set by

$$H(r_E) = \log_2 |E|, \quad r_E(x) = \begin{cases} 1 & \text{when } x \in E \\ 0 & \text{when } x \notin E \end{cases} \tag{8}$$

with $|E|$ the cardinality of the set E. Assume two sets X and Y and the set $R \subseteq X \times Y$ that describe possible alternatives in some situation of interest. Then following relations between the marginal, joint and conditional Hartley measures hold:

$$H(X|Y) = H(X \times Y) - H(Y) \tag{9}$$

$$H(Y|X) = H(X \times Y) - H(X) \tag{10}$$

If possible alternatives from X do not depend on selections from Y, and vice versa, then $R = X \times Y$ and the sets R_X and R_Y (the projections of R on X and Y, respectively) are called noninteractive:

$$H(X|Y) = H(X) \tag{11}$$

$$H(Y|X) = H(Y) \tag{12}$$

$$H(X \times Y) = H(X) + H(Y) \tag{13}$$

In the general case of unknown interactivity, these equalities become inequalities $\leq$. To indicate the strength of constraint between possible alternatives in sets X and Y, the information transmission is defined as

$$T_H(X, Y) = H(X) + H(Y) - H(X \times Y) \tag{14}$$

When the sets are noninteractive, $T_H(X, Y) = 0$; otherwise, it is positive. Its maximum value is obtained if $H(X|Y) = H(Y|X) = 0$, or in other words, if the value of X is specified, only one value for Y is possible. This indicator can be considered a measure of dependence in this paper.

3.2. Graded possibilities

In the framework of graded possibilities a value between zero and one is assigned to every singleton of a set X by the basic possibility function $r(x)$, with the requirement of possibilistic normalization to assign at least to one x the value 1. The possibility assigned to a subset A of X is determined by

$$Pos(A) = \max_{x \in A} \{r(x)\} \tag{15}$$

Based on a joint possibility function $r(x, y)$ defined over $X \times Y$ the marginal possibility functions are defined as

$$r_X(x) = \max_{y \in Y} \{r(x, y)\}, \quad \forall x \in X \tag{16}$$

$$r_Y(y) = \max_{x \in X} \{r(x, y)\}, \quad \forall y \in Y \tag{17}$$

If it is known that variables X and Y do not interact, then the joint possibility function r is defined by

$$r(x, y) = \min\{r_X(x), r_Y(y)\} \tag{18}$$

Among all joint possibility profiles that are consistent with the given marginal possibilities, this one based on the assumption of noninteractive marginals, is the largest one. The definition of conditional possibilities can go many ways (Fonck, 2006). We follow the definition by (Hisdal, 1978):

$$r_{X|Y} = \begin{cases} r(x, y) & \text{when } r_Y(y) > r(x, y) \\ 1 & \text{when } r_Y(y) = r(x, y) \end{cases} \tag{19}$$

$$r_{Y|X} = \begin{cases} r(x, y) & \text{when } r_X(x) > r(x, y) \\ 1 & \text{when } r_X(x) = r(x, y) \end{cases} \tag{20}$$

To define possibilistic independence based on this definition of conditional possibilities, one can again go many ways. The key is to compare $r_X(x)$ and $r_{X|Y}(x|y)$, this can be done in at least three ways as suggested in (de Campos and Huete, 1999). Possibilistic independence can be defined based on: equality $r_X(x) = r_{X|Y}(x|y)$ (not modifying information), inequality $r_X(x) \leq r_{X|Y}(x|y)$ (not gaining information) or similarity $r_X(x) \simeq r_{X|Y}(x|y)$ (obtaining similar information, but specification of the similarity measure is needed). Here, as in (Klir, 2006), the equality operator is adopted. With such a definition possibilistic independence implies possibilistic noninteraction, but not the other way around.

This definition of noninteraction or independence does not give us a measure of interaction or dependence. First attempts to come up with such a measure are apparently found in (Fuller and Majlender, 2004).

4. Spatial uncertainties and dependence measures

As mentioned in the introduction, the modeling of a spatial uncertainty calls for a dependence measure. In particular, one needs a dependence description in function of the distance between points.

4.1. Probabilistic spatial uncertainty and correlation length

In the probabilistic framework, the concept of a random field (Vanmarcke, 1993) is well developed. In its application the crucial element is the specification of the correlation structure. For homogeneous random fields, this correlation structure describes the value of the correlation as a function of the distance between two points. A crucial parameter in this function is the correlation length as made clear in the illustrative sensitivity study (Charmpis et al., 2007). The parameter largely dominates the discretization of a random field. For an overview of discretization methods applicable to finite element analysis, the reader is referred to the excellent report by (Sudret And Der Kiureghian, 2000). Three groups of discritization are identified: point discretization, average discretization and series expansion methods.

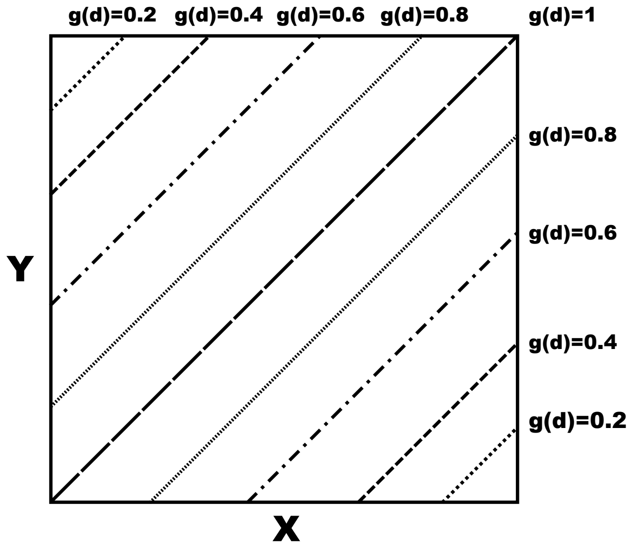

Figure 2. The possible values of $R \subseteq X \times Y$ as a function of $g(d)$. The limits on the possible values are given by identically dashed lines.

4.2. Non-probabilistic spatial uncertainty and domain of influence

Apparently a link between a crisp possibilistic measure of dependence and spatial uncertainty is not yet formulated in literature. Our suggestion consists in specifying a function $g(d) : d \to \lfloor -\infty, 1 \rceil$ with d the (non-negative) distance between two points in a model. To every value $g(d)$ corresponds a set $R \subseteq X \times Y$ of possible values. For $g(d) \leq 0$ the set $R = X \times Y$, for $g(d) = 1$ the set R reduces to the single line $X = Y$. In other words, the information transmission becomes maximal and the possible alternatives for X given Y reduce to one, if the distance between X and Y reduces to zero. All this leads to the following conditions on $g(d)$:

$$\begin{cases} g(0) = 1 \\ g(d_1) > g(d_2) \quad \text{for } d_1 < d_2 \end{cases} \tag{21}$$

A simple example of such a function $g(d)$ is

$$g(d) = 1 - \frac{d}{a} \tag{22}$$

with $a > 0$ a parameter specifying the domain of influence. If $d < a$ then $R \subset X \times Y$, for $d > a$ $R = X \times Y$.

For values $0 < g(d) < 1$ the domain of possible values R can take many shapes, our focus is restricted to two cases. The first is a shape similar to the diagonal copula discussed above. Figure 2 illustrates the concept, where the limits on the possible values are given by identically dashed lines. A clear link between this representation and bounding the spatial derivative $0 < |X'| \leq z$ of a model parameter (in addition to bounding its value $l \leq X \leq u$) can be established. Let two locations in a model seperated by a distance d be given index 1 and 2. Given a value for x_1, the value of x_2 is bounded by $x_1 - zd$ and $x_1 + zd$ as long as $x_1 - zd > l$ and $x_1 + zd < u$. These bounds can be directly related to the model for $g(d)$ in Eq. (22) with

$a = (l - u)/z$. An example where the bounds on the spatial derivative of an uncertain model parameter are used to describe the spatial dependence can be found in (Ben-Haim and Elishakoff, 1990).

The second shape is a nesting of ellipses for $0 < g(d) < 1$, that degenerate to a square for $g(d) = 0$ and degenerate to a line for $g(d) = 1$. The basis of a practical method to calculate the ellipses based on experimentally measured spatial data can be found in (Zhu et al., 1996).

5. Bounding dependence measures for spatial uncertainties

The actual value of the correlation length or the parameter a in Eq. (22) is seldomly known. For this reason, it is suggested to treat them as intervals. For a study of the influence of an interval correlation length on the series expansion of a random field, the reader is referred to (Verhaeghe et al., 2011). In the example below a similar analysis is performed for a point discretization of a random field. After the discretization is fixed, the interval on the correlation length, actually results in intervals on the correlations between all the discretization points.

For the non-probabilistic case, after fixing the discretization, the interval on a results in a set of domains of influence. Points that are not influencing eachother for small values of a, become important to each other with increasing a. Depending on the studied output, its interval can increase or decrease in size with increasing a.

6. Numerical example

In this section, the influence of bounds on a dependence measure for both a probabilistic and a non-probabilistic analysis is studied in the context of a numerical example.

6.1. Reliability of beam with random flexibility

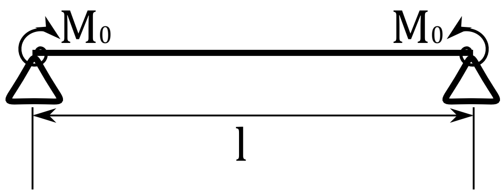

Figure 3. A simply supported beam of length l, loaded with constant moment M_0

A beam of length l is simply supported at its both ends and loaded with a constant moment M_0 (see Figure 3). The flexibility $h(x)$ is characterized by

$$h(x) = H_1\phi_1(x) + H_2\phi_2(x) \tag{23}$$

where $\phi_1(x)$, takes value 1 for the left half ($0 \leq x < 0.5l$) and has value 0 for the right half of the beam. Conversely, the function $\phi_2(x)$ takes value 1 for the right half of the beam and 0 for the left half. The amplitudes H_1 and H_2 are each a uniformly distributed random variable. As such, the flexibility is modelled by a very coarse point discretisation (only the two points $x = 0.25l$ and $x = 0.75l$ are considered) of the random field for the flexibility.

The performance of the beam in this case is determined by the displacement difference between two points symmetrically located on both sides of its mid-point. The displacement is calculated using

$$\begin{aligned} W(x) &= M_0 \int_0^x \int_o^v h(u)dudv - M_0 \frac{x}{l} \int_0^l \int_0^v h(u)dudv \\ &= M_0 \left[H_1 \int_0^x \int_o^v \phi_1(u)dudv + H_2 \int_0^x \int_o^v \phi_2(u)dudv \right. \\ & \left. - \frac{x}{l} \left(H_1 \int_0^l \int_0^v \phi_1(u)dudv + H_2 \int_0^l \int_0^v \phi_2(u)dudv \right) \right] \end{aligned} \tag{24}$$

The displacement difference between points x_1 and x_2 is thus found by

$$\Delta W_{x_1,x_2} = W(x_1) - W(x_2) \tag{25}$$

Defining $A(x) = \int_0^x \int_o^v \phi_1(u)dudv$ and $B(x) = \int_0^x \int_o^v \phi_2(u)dudv$, the reliability can be calculated as:

$$\begin{aligned} R &= P\left(|\Delta W_{x_1,x_2}| \leq \Delta W_{specified}\right) \\ &= P\left(\left|M_0 \left[H_1 \left(A(x_1) - \frac{x_1}{l} A(l)\right) + H_2 \left(B(x_1) - \frac{x_1}{l} B(l)\right)\right.\right.\right. \\ & \qquad \left.\left. - H_1 \left(A(x_2) - \frac{x_2}{l} A(l)\right) - H_2 \left(B(x_2) - \frac{x_2}{l} B(l)\right)\right] \right| \leq \Delta W_{specified}\Big) \\ &= P\left(\left|M_0 \left[H_1 \left(A(x_1) - A(x_2) - \frac{x_1}{l} A(l) + \frac{x_2}{l} A(l)\right)\right.\right.\right. \\ & \qquad \left.\left. + H_2 \left(B(x_1) - B(x_2) - \frac{x_1}{l} B(l) + \ B(l)\right)\right] \right| \leq \Delta W_{specified}\Big) \\ &= P\left(\left|M_0 \left[H_1 A^* + H_2 B^*\right]\right| \leq \Delta W_{specified}\right) \\ &= \int_0^{\overline{H_1}} \int_{\frac{-\Delta W_{specified}/M_0 - \xi_1 A^*}{B^*}}^{\frac{\Delta W_{specified}/M_0 - \xi_1 A^*}{B^*}} f_H(\xi_1, \xi_2) d\xi_2 d\xi_1 \\ &= \int_0^{\overline{H_2}} \int_{\frac{-\Delta W_{specified}/M_0 - \xi_2 B^*}{A^*}}^{\frac{\Delta W_{specified}/M_0 - \xi_2 B^*}{A^*}} f_H(\xi_1, \xi_2) d\xi_1 d\xi_2 \end{aligned} \tag{26}$$

with A^* and B^* the weights of the random variables due to the integrals of $\phi_1(x)$ and $\phi_2(x)$ respectively; $\overline{H_1}$ and $\overline{H_2}$ the upper bounds of the random variables.

Let us assume $l = 1$, $M_0 = 1$, both H_1 and H_2 uniform on $\lfloor 0.95, 1.05 \rceil$, $x_1 = 0.4$, $x_2 = 0.6$ and $\Delta W_{specified} = 0.001$. The joint density $f_H(\xi_1, \xi_2)$ is chosen equal to the diagonal band copula discussed above. The density is characterised by the parameter $b = 1 - a$. For $b = 1$, the copula represents independence. For $b = 0$, the copula represents perfect positive dependence. The density is described by

$$C_a(u_1, u_2) = \frac{1}{2(1-a)} \left(I_{[a-1,1-a]}(u - v) + I_{[0,1-a]}(u + v) + I_{[1+a,2]}(u + v) \right) \tag{27}$$

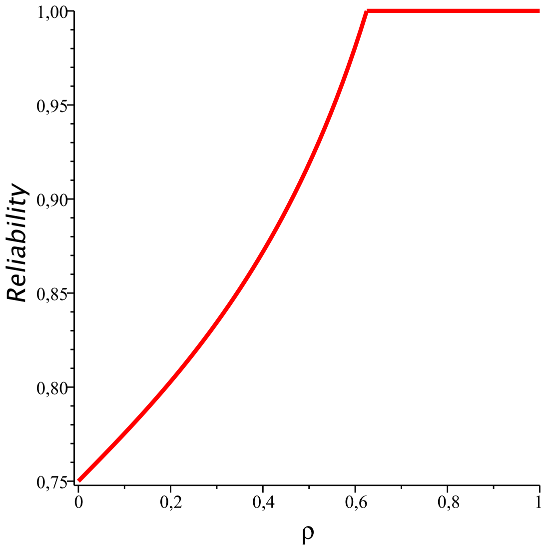

Figure 4. The reliability of the beam as a function of the correlation

With I_A, the indicator function of A. Parameter b is a function of the (rank) correlation as found in (Kurowicka and Cooke, 2006):

$$b = 1 - a = \frac{2}{3} - \frac{4}{3}\sin\left(\frac{1}{3}\arcsin\left[\frac{27}{16}\rho - \frac{11}{16}\right]\right) \tag{28}$$

Figure 4 shows the reliability for the correlation varying between 0 and 1. For a higher correlation coefficient the reliability tends to 1. If both variables H_1 and H_2 are completely independent, the reliability is only 0.75. If one is able to bound the correlation between the two variables, the corresponding bounds on the reliability can be found from the figure.

6.2. Bounds on the displacement difference of the beam with interval flexibility

The same beam as above is considered, but now H_1 and H_2 are intervals between 0.95 and 1.05. The quantity of interest is again the displacement difference between the same two points symmetrically located on both sides of the mid-point. The dependence between the two intervals is characterised by a joint set as illustrated in Figure 2, with $g(d)$ as in Eq. (22) with $d = 0.5l$, the distance between the two discretization points on the beam and $a \in \lfloor 0.5l, 2l \rceil$. This interval description for a results in a situation where one of the two discretization points is just on the boundary of the domain of influence of the other point when $a = 0.5l$ and $g(0.5l) = 0$. In the other extreme case, the other point resides on the $g(05l) = 0.75$ (with $a = 2$) limit in the domain of influence.

The upper bound for the absolute value of the displacement difference as a function of a is shown in Figure 5. The lower bound is always 0.

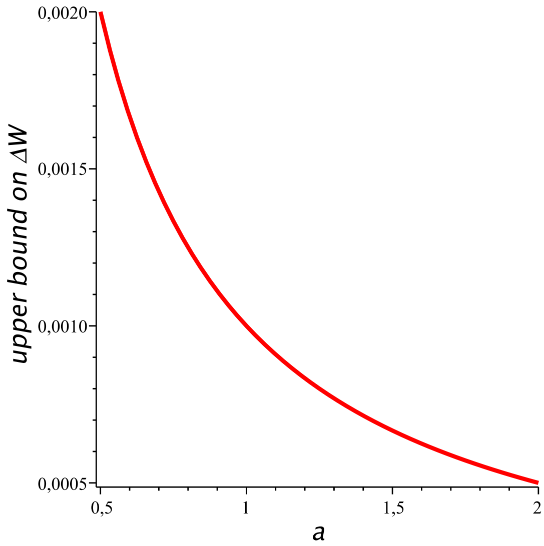

Figure 5. The upper bound on the displacement difference

7. Conclusion

The paper presents a review of dependence measures for both probabilistic and non-probabilistic descriptions of uncertainty. The link with numerically modeling a spatial uncertainty is established based on the functional relation between the dependence measure and a distance measure in a numerical model. The additional uncertainty related to this functional relationship is treated by representing the reference distance (i.e. correlation length or domain of influence) as an interval. The procedure is illustrated on a numerical example.

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