Dynamic Response of Beams to Interval Load

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Abstract. Parameters of mathematical models are most often represented by real numbers, while in practice it is impossible or at least very difficult to get reliable information about their exact values. Hence, it is unreasonable to take point data for that may lead to incorrect results, which is not welcome especially when inaccuracy cannot be neglected. Depending on available information, one can use different ways of modelling of uncertainty. Interval computing plays an important role in this field, because very often the only available information are lower and upper bounds on a physical quantity. This paper focuses on a transient dynamic analysis of a beam with uncertain parameters. Finite difference and finite element methods are used to solve partial differential equation which represents the model for the motion of a straight elastic beam. In order to compute the time-history response of the beam under uncertainty, interval dynamic beam equations are solved using Search method, Gradient method, Taylor method, adaptive Taylor method, direct optimisation and Direct method for solving parametric interval linear systems. The applicability, i.e. effectiveness and accuracy, of those methods is illustrated through solution of beams with interval value of modulus of elasticity and mass density and subjected to interval dynamic loading.

Keywords: Euler-Bernoulli beam, Dynamic response, Interval arithmetic, Search method, Gradient method, Taylor method, adaptive Taylor method, Direct method, Direct optimisation.

1. Introduction

Airplane wings, high-rise buildings and suspension bridges are just some of the mechanical and structural examples where vibration analysis of beams is essential for the safe design. Safety issues are the greatest concern of structural engineering as the design and construction of secure and safe structures can prevent disasters like the collapse of Tacoma Narrow Bridge November 7, 1940, just few month after it was finished. This was probably the most dramatic failure in bridge engineering history. Safety studies in structural engineering are supposed to prevent failure during the lifetime of a structure.

Constantly increasing computational capabilities allow for detailed numerical models of structural systems. However, those models are built, inter alia, on a number of model parameters subject to uncertainty.

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The use of models that include the uncertainty, which is central to reliability/risk analysis of engineering systems, is of great importance for a design engineer.

Uncertainty of structural parameters is mainly due to the scarcity or lack of data which may be resulted from manufacturing/construction tolerances or caused by progressive deterioration of concrete and corrosion of steel. In engineering applications, uncertainty also exists in determining external loads. To make a decision based on an inexact data say some parameter $\tilde{p}$, a measurement error $\Delta p = |\tilde{p} - p|$ must known at least. Very often, the only available information about the error is its upper bound $\Delta p \leqslant \Delta$. In this case, once the measurement $\tilde{p}$ is obtained, one can conclude that the possible values $\tilde{p} + \Delta p$ form an interval $\boldsymbol{p} = [\tilde{p} - \Delta, \tilde{p} + \Delta]$ which is guaranteed to contain the exact value p of the parameter. Once interval quantities are introduced, they must be handled appropriately to obtain the result which is guaranteed to contain the exact solution.

Though interval arithmetic was introduced by Moore (Moore, 1966) already in 1966, the application of interval concepts to structural analysis is more recent. Some important advances on reliability-based design and modelling of uncertainty when data is limited were made during last years. Structural analysis using interval variables has been used by several researchers to incorporate uncertainty into structural analysis ((Köylüoglu et al., 1995), (Nakagiri and Yoshikawa, 1996), (Rao and Sawyer, 1995), (Rao and Berke, 1997), (Rao and Chen, 1998), (Mullen and Muhanna, 2001), (Neumaier and Pownuk, 2007), (Skalna, Pownuk and Rama Rao, 2008)).

In this paper, the problem of vibrations of an Euler-Bernoulli beam with interval material properties subjected to interval load is considered. Two different approaches are employed to obtain beam deflection in time. In the first approach, the Euler-Bernoulli equation governing the behaviour of the beam is descretized in space and time. The beam bending in the respective time step is obtained by solving a system of equations with coefficient depending on interval parameters. Several methods are used for this purpose. Search method, Gradient method ((Skalna, Pownuk and Rama Rao, 2008)), Taylor method and adaptive Taylor method (Pownuk, 2011) utilise the fact that in many structural engineering problems relation between the solution and uncertain parameters is monotone. In such a case, the extreme values of a solution are attained at respective endpoints of given intervals. Monotonicity can be verified by using Taylor series or an interval method (Hansen, 1992). Methods exploiting monotonicity tests are useful for solving large scale problems, but they may underestimate. When monotonicity is not assumed, the solution can be obtained using methods for solving parametric interval linear systems (Skalna, 2010)). Those methods give guaranteed enclosures, but their usage is limited e.g. by the amount of uncertainty. In the second approach, the Finite Element Method is a starting point for considerations. The Wilson-θ method and optimisation approach are used for the solution of the problem (Rama Rao, Pownuk and Vandewalle, 2010).

The paper has the following structure. In Sections 3 and 6, the considered problem is described in terms of the mathematical theory. Section 4 describes the discretization of the problem in time and space. Section 5 and 6.1 are devoted to the methods for solving interval linear systems obtained from the discretization of the Euler-Bernoulli equation. Numerical examples are given in Section 7. The paper ends with concluding remarks.

2. Interval uncertainty

If only very limited knowledge about the value of some structural parameter p_i is available, then this value can be conveniently described by an interval number in the following way:

$$p_i \in [\tilde{p}_i - \Delta p_i, \tilde{p}_i + \Delta p_i] = [\underline{p}_i, \overline{p}_i] = \boldsymbol{p}_i, \tag{1}$$

where $\tilde{p}_i$ can be considered as an approximation of the true value of p_i and Δp_i as an approximation error.

Now, if some output quantity y is related to parameters p by a known relation $y = f(p)$, then the calculation of the result, assuming p vary within $\boldsymbol{p}$, is numerically equivalent to finding the following solution set:

$$y_S = \{y \; : \; y = f(p), p \in \boldsymbol{p}\}. \tag{2}$$

The outcome of the interval analysis here is expressed as a set y_S of possible solutions as, in general, it cannot be described exactly by an interval or hypercube. The correct interpretation of this expression is that the set y_S contains all vectors y that are obtained from applying the function f on all possible vectors p within the interval vector $\boldsymbol{p}$.

An exact description of the solution set y_S is often extremely difficult to find. Therefore, usually an interval vector $\boldsymbol{x}^* \ni y_S$, called outer solution/enclosure, is computed instead and the goal is $\boldsymbol{x}^*$ to be as narrow as possible. The tightest interval vector containing y_S is called *hull solution* (or simply a *hull*). One can also calculate inner solution/approximation which is defined as an interval vector which is included in the hull. They are usually obtained using the "straightforward" interval arithmetic. However, this usually leads to large overestimation due to the so-called *dependency problem*. Keeping track of how intermediate results on input data may decrease excess with. This idea was successfully implemented in several approaches, e.g. affine arithmetic (Comba and Stolfi, 1993).

3. Forced vibration of a beam

Forced vibration of a beam is governed by Euler-Bernoulli equation (Ciarlet, 1997).

$$\frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 w}{\partial x^2}\right) = q - \rho A\frac{\partial^2 w}{\partial t^2}, \tag{3}$$

where E is the elastic modulus, J is the second area moment, A is the cross-sectional area, ρ is mass density of the material of the beam and q is an external load. The model (3) where the displacement w depends only on one-dimensional spatial variable x and time t is obtained upon the use of Hookes law and other simplifying assumptions. This model is a valid approximation for thin beams under small transverse deformations. As a good rule-of-thumb, 'small' is defined as deflections that are at least ten times smaller than beam thickness.

For an uniform beam (EJ is constant), Eq. (3) reduces to

$$EJ\frac{\partial^4 w}{\partial x^4} = q - \rho A\frac{\partial^2 w}{\partial t^2}. \tag{4}$$

Because vibration is an initial-boundary value problem, therefore both initial and boundary conditions are required to obtain a unique solution $w(x, t)$. Since the equation involves second order derivative with respect to time and fourth derivative with respect to a space coordinate, thus four boundary and two initial conditions are necessary:

$$\begin{cases} w(0,t)=0 \\ w(L,t)=0 \\ \dfrac{\partial^2 w}{\partial x^2}(0,t)=0 \\ \dfrac{\partial^2 w}{\partial x^2}(L,t)=0 \end{cases}, \quad \begin{cases} w(x,0)=w_0(x) \\ v(x,0)=\frac{\partial w}{\partial t}(x,0)=v_0(x) \end{cases} \tag{5}$$

Endpoint displacements are equal to zero, which can be written as $w(0,t)=w(L,t)=0$ for $t\in[0,T]$. Because bending moments at both endpoints are equal to zero, therefore $M(0,t)=EJ\dfrac{\partial^2 w}{\partial x^2}(0,t)$ and $M(L,t)=EJ\dfrac{\partial^2 w}{\partial x^2}(L,t)$, and consequently $\dfrac{\partial^2 w}{\partial x^2}(0,t)=\dfrac{\partial^2 w}{\partial x^2}(L,t)=0$ for $t\in[0,T]$. For $t=0$, both displacement and velocity are equal to zero and thus $w_0(x)=0$, $v_0(x)=0$ for $x\in[0,L]$.

4. Implicit Finite Difference Discretization

In this paper implicit Finite Difference Method has been applied to the problem of dynamic beam vibrations (Ciarlet, 1990). Discretization of Eq. (4) is performed at point $(i,j+1)$:

$$\left(EJ\frac{\partial^4 w}{\partial x^4}\right)_{i,j+1}=q_{i,j+1}-\left(\rho A\frac{\partial^2 w}{\partial t^2}\right)_{i,j+1} \tag{6}$$

which leads to the finite difference equation:

$$\begin{aligned} &E_{i,j+1}J_{i,j+1}\frac{w_{i+2,j+1}-4w_{i+1,j+1}+6w_{i,j+1}-4w_{i-1,j+1}+w_{i-2,j+1}}{\Delta x^4}+\frac{\rho_{i,j+1}A_{i,j+1}}{\Delta t^2}w_{i,j+1}= \\ &=q_{i,j+1}-\rho_{i,j+1}A_{i,j+1}\frac{2w_{i,j}-w_{i,j-1}}{\Delta t^2} \end{aligned} \tag{7}$$

Similarly, it is possible to discretize initial and boundary conditions. Finally, one obtains:

$$\begin{cases} w_{0,j+1}=0 \\ w_{0,j+1}-2w_{1,j+1}+w_{2,j+1}=0 \\ E_{i,j+1}J_{i,j+1}\dfrac{w_{i+2,j+1}-4w_{i+1,j+1}+6w_{i,j+1}-4w_{i-1,j+1}+w_{i-2,j+1}}{\Delta x^4}+\dfrac{\rho_{i,j+1}A_{i,j+1}}{\Delta t^2}w_{i,j+1}= \\ =q_{i,j+1}-\rho_{i,j+1}A_{i,j+1}\dfrac{2w_{i,j}-w_{i,j-1}}{\Delta t^2} \\ w_{n-2,j+1}-2w_{n-1,j+1}+w_{n,j+1}=0 \\ w_{n,j+1}=0 \\ w_{i,0}=w_i^* \\ w_{i,1}=w_{i,0}+v_i^*\Delta t. \end{cases} \tag{8}$$

It is important to note that $w_{i,j+1}=w(p_{i,j+1})$ where $p_{i,j+1}=(E_{i,j+1},\rho_{i,j+1},q_{i,j+1})$. Discretization reduces the problem of computing the dynamic response of a beam to the problem of solving a sequence of parametric linear systems. Assuming the uncertainty of the parameters, an sequence of parametric interval linear systems must be solved. In order to increase the accuracy of the FDM, finite difference scheme of order higher than 3 has been applied for the time step.

$$\left(\frac{\partial^2 w}{\partial t^2}\right)_{i,j} \approx \frac{2w_{i,j} - 5w_{i,j-1} + 4w_{i,j-2} - w_{i,j-3}}{\Delta t^2} \tag{9}$$

5. Methods for solving parametric linear systems

Apart from the diversity caused by the nature of the numerical problem at hand, a clear distinction can be made between fundamental approaches for tackling the interval uncertainty. The interval arithmetic strategy approaches the exact hypercubic circumscription of the interval result from outside. It is based on the calculation of guaranteed outer bounds. The global optimisation approach on the other hand calculates an inner approximation. The interval arithmetic based methods proves to be computationally less expensive than the approximate method, it very often results in a huge overestimation of the actual interval result, due to the dependency problem. On the other hand, optimisation based approaches, though computationally expensive and time-consuming, provide an acceptable solution for practical engineering problems.

5.1. Interval solution as a function of uncertain parameters

Each interval solution is in fact a function of some specific combinations of the parameters:

$$\underline{w}_{i,j} = w_{i,j}(p_{i,j}^{\min}), \qquad \overline{w}_{i,j} = w_{i,j}(p_{i,j}^{\max}). \tag{10}$$

In the continuous case, one can write

$$\begin{aligned}\underline{w}(x,t) &= w(x,t,p^{\min}(x,t)),\\ \overline{w}(x,t) &= w(x,t,p^{\max}(x,t)).\end{aligned} \tag{11}$$

In some situations, the interval solution depends only on one combination of parameters for some domain $D_\alpha \subseteq [0,L] \times [0,T]$

$$\underline{w}(x,t) = w(x,t,p_\alpha^{\min}), \qquad \overline{w}(x,t) = w(x,t,p_\alpha^{\max}). \tag{12}$$

In such cases it is possible to calculate the interval solution exactly by using finite number of combinations of the parameters $p_1^{\min}$, $p_1^{\max}$,...,$p_\alpha^{\min}$, $p_\alpha^{\max}$,...,$p_q^{\min}$, $p_q^{\max}$ where $D_1 \cup ... \cup D_q = [0,L] \times [0,T]$.

5.1.1. *Search method*

In order to find the interval solution the Search method is applied. The method relies on solving parametric linear systems of equations corresponding to the specific combinations of the parameters. That is, each interval parameter p_i is replaced by the set of discrete points $p_{i1}, \ldots, p_{ik}$:

$$\boldsymbol{p}_i \approx \{p_{i1}, \ldots, p_{ik}\}. \tag{13}$$

A multidimensional interval $\boldsymbol{p} = [\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m]$ is approximated by the discrete set of points:

$$\boldsymbol{p} = [\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m] \approx \{(p_{1,i_1}, \ldots, p_{m,i_k}) : 0 \leqslant i_1, \ldots, i_k \leqslant k\} = \mathcal{P}_{m,k}. \tag{14}$$

Number of elements in the set (14) equals k^m where m is the number of interval parameters and k is the number of intermediate points in each interval p_i (for $k = 2$ the method reduces to the endpoints combination method). Then, an interval solution can be calculated in the following way:

$$\underline{w}_{i,j} \approx \underline{w}_{i,j}^{search} = \min\{w_{i,j}(p_{1,i_1}, \ldots, p_{m,i_k}) : (p_{1,i_1}, \ldots, p_{m,i_k}) \in \mathcal{P}_{m,k}\}, \tag{15}$$

$$\overline{w}_{i,j} \approx \overline{w}_{i,j}^{search} = \max\{w_{i,j}(p_{1,i_1}, \ldots, p_{m,i_k}) : (p_{1,i_1}, \ldots, p_{m,i_k}) \in \mathcal{P}_{m,k}\}. \tag{16}$$

The search method allows as well finding approximate values of $p_{i,j}^{\min,search}, p_{i,j}^{\max,search}$.

$$\underline{w}_{i,j} \approx \underline{w}_{i,j}^{search} = w_{i,j}(p_{i,j}^{\min,search}), \overline{w}_{i,j} \approx \overline{w}_{i,j}^{search} = w_{i,j}(p_{i,j}^{\max,search}) \tag{17}$$

According to numerical experiments

$$p_{i,j}^{\min} \approx p_{i,j}^{\min,search}, p_{i,j}^{\max} \approx p_{i,j}^{\max,search} \tag{18}$$

which means that the Search method can find approximate or exact values of $p_{i,j}^{\min}$ and $p_{i,j}^{\max}$.

5.1.2. *Gradient method*

The value of $p^{\min}(x,t)$ and $p^{\max}(x,t)$ can be found as well by solving respectively the following minimisation and maximisation problems

$$\begin{aligned} &p^{\min}(x,t) = \arg\min_p w(x,t,p), \\ &p^{\max}(x,t) = \arg\max_p w(x,t,p), \\ &\text{s.t.} \begin{cases} EJ\dfrac{\partial^4 w}{\partial x^4} = q - \rho A \dfrac{\partial^2 w}{\partial t^2} \\ w(0,t) = 0 \\ w(L,t) = 0 \\ \dfrac{\partial^2 w}{\partial x^2}(0,t) = 0 \\ \dfrac{\partial^2 w}{\partial x^2}(L,t) = 0 \\ w(x,0) = w_0(x) \\ v(x,0) = \dfrac{\partial w}{\partial t}(x,0) = v_0(x) \\ p \in \boldsymbol{p} \end{cases} \end{aligned} \tag{19}$$

Solutions $w_{i,j}$ are functions of uncertain parameters $w_{i,j} = w_{i,j}(p)$. If the function $w_{i,j} = w_{i,j}(p)$ is monotone, then $p_{i,j}^{\min}$ and $p_{i,j}^{\max}$ can be calculated as:

$$p_{i,j,k}^{\min,gradient} = \underline{p}_k, \;\; if \;\; \frac{\partial w_{i,j}}{\partial p_k} < 0 \;\; else \;\; p_{i,j,k}^{\min,gradient} = \overline{p}_k, \tag{20}$$

$$p_{i,j,k}^{\max,gradient} = \overline{p}_k, \;\; if \;\; \frac{\partial w_{i,j}}{\partial p_k} \geq 0 \;\; else \;\; p_{i,j,k}^{\max,gradient} = \underline{p}_k, \tag{21}$$

where $\frac{\partial w_{i,j}}{\partial p_k}$ are partial derivatives with respect to all uncertain parameters. Thus obtained combinations of endpoints can be utilised for calculation of upper and lower bounds of the solution

$$\underline{w}_{i,j} \approx \underline{w}_{i,j}^{gradient} = w_{i,j}(p_{i,j}^{\min,gradient}), \qquad \overline{w}_{i,j} \approx \overline{w}_{i,j}^{gradient} = w_{i,j}(p_{i,j}^{\max,gradient}). \tag{22}$$

5.1.3. *Taylor method*

The interval solution can be also calculated by using first order Taylor model:

$$w_{i,j}(p) \approx T_{i,j}^{(1)}(p) = w_{i,j}(p_0) + \sum_k \frac{\partial w_{i,j}}{\partial p_k}(p_0) \cdot (p_k - p_0). \tag{23}$$

In this approach the interval solution can be calculated in the following way

$$\underline{w}_{i,j} \approx \underline{w}_{i,j}^{Taylor} = w_{i,j}(p_0) - \sum_k \left| \frac{\partial w_{i,j}}{\partial p_k}(p_0) \right| \cdot \Delta p_k, \tag{24}$$

$$\overline{w}_{i,j} \approx \overline{w}_{i,j}^{Taylor} = w_{i,j}(p_0) + \sum_k \left| \frac{\partial w_{i,j}}{\partial p_k}(p_0) \right| \cdot \Delta p_k, \tag{25}$$

The result of the Taylor method can be calculated as well by using endpoint combinations and Taylor polynomial

$$\underline{w}_{i,j}^{Taylor} = T_{i,j}^{(1)}(p_{i,j}^{\min,gradient}), \quad \overline{w}_{i,j}^{Taylor} = T_{i,j}^{(1)}(p_{i,j}^{\max,gradient}). \tag{26}$$

5.1.4. *Adaptive Taylor approximation*

It is possible to increase the accuracy of the Taylor method results by using *adaptive approximation* (Pownuk, 2011). It is necessary to calculate all different combinations of parameters $L_1 = \{p^{*,1}, ..., p^{*,n_1}\}$ in the sets $p_{i,j}^{\min,gradient}$ and $p_{i,j}^{\max,gradient}$. For each combination $p^{(*,k)}$ from the list L_1 it is necessary to find a point solution $w^{(k)} = w(p^{*,k})$.

$$\underline{w}_{i,j}^{(1)} \approx min\{w_{i,j}^{(1)}, ..., w_{i,j}^{(n_1)}\}, \quad \overline{w}_{i,j}^{(1)} \approx max\{w_{i,j}^{(1)}, ..., w_{i,j}^{(n_1)}\} \tag{27}$$

For $\underline{w}_{i,j}^{(1)}$ and $\overline{w}_{i,j}^{(1)}$ it is necessary to calculate new values of $p_{i,j}^{\min,gradient,1}$ and $p_{i,j}^{\max,gradient,1}$. In the sets $p_{i,j}^{\min,gradient,1}$ and $p_{i,j}^{\max,gradient,1}$ it is necessary to find new combinations of parameters and add to the list L. New list will be denoted as L_2 and calculate new values of upper and lower bound from the formula (27). Calculations will be stopped if no new combinations of parameters will be found in the next iteration i.e. $L_i = L_{i+1}$.

5.2. DIRECT METHOD

To verify the results obtained using the approximate methods described in the previous sections, the direct method (DM) (Skalna, 2010) for solving parametric interval linear systems is applied to the problem.

5.2.1. *Iterative approach*
The system (8) can be written in matrix form as:

$$Kw^{j+1} = Q^j + M_2 w^j + M_1 w^{j-1},\ (j = 2, \dots, n_t) \tag{28}$$

with

$$Q^j = \begin{bmatrix} 0 \\ 0 \\ q_{2,j+1} \\ q_{3,j+1} \\ \vdots \\ q_{n-2,j+1} \\ 0 \\ 0 \end{bmatrix},\ M_1 = \mu_2 \begin{bmatrix} 0 & 0 & & & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ & & -1 & & & \\ & & & \ddots & & \\ & & & & -1 & \\ 0 & 0 & & & 0 & \\ 0 & 0 & & & & 0 \end{bmatrix},\ M_2 = \mu_2 \begin{bmatrix} 0 & 0 & & & 0 & 0 \\ 0 & 0 & & & 0 & 0 \\ & & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & \\ 0 & 0 & & & 0 & \\ 0 & 0 & & & & 0 \end{bmatrix}, \tag{29}$$

and

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & -4\mu_1 & 6\mu_1+\mu_2 & -4\mu_1 & \mu_1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & -4\mu_1 & 6\mu_1+\mu_2 & -4\mu_1 & \mu_1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \mu_1 & -4\mu_1 & 6\mu_1+\mu_2 & -4\mu_1 & \mu_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{30}$$

where $\mu_1 = \frac{EJ}{\Delta x^4}$, $\mu_2 = \frac{\rho A}{\Delta t^2}$. As can be seen, the coefficients of the system (28) are polynomial functions of the parameters E, ρ and q. Thus, the Direct Method can be applied. The interval result obtained in a given iteration step enters into the right-hand of the system to be solved in the next step. The iterative approach is quite efficient, however the overestimation of the result grows in successive iteration steps. This is mainly due to the so-called *dependency problem*. Namely, starting from the second iteration, the dependency in the right hand vector is lost since the solution enters the right-hand as an interval vector and not the affine one. Therefore, in what follows a non-iterative approach is considered.

5.2.2. *Non-iterative formulation*
Consider the linear system (28). Taking into account boundary conditions, one obtains:

$$\begin{cases} w^0 = 0 \\ w^1 = 0 \\ -M_1 w^0 - M_2 w^1 + K w^2 = Q^1 \\ \dots \\ -M_1 w^{n_t-2} - M_2 w^{n_t-1} + K w^{n_t} = Q^{n_t-1} \end{cases}, \tag{31}$$

where $w^j = [w_{0,j}, w_{1,j}, \ldots, w_{n_x,j}]$. Finally, the following parametric linear system is obtained:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ -M_1 & -M_2 & K & 0 & \ldots & 0 \\ 0 & -M_1 & -M_2 & K & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -M_1 & -M_2 & K \end{bmatrix} \begin{bmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \\ \vdots \\ w^{n_t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Q^1 \\ Q^2 \\ \vdots \\ Q^{n_t-1} \end{bmatrix} \tag{32}$$

The non-iterative approach is free from the accumulation error problem described above, but suffers from the efficiency problem as a very large system (of size $(n_x + 1)n_t$) must be solved.

6. Wilson-θ method

Consider a discrete structural system with multi-degree of freedom (MDOF) described by equation

$$M\ddot{w} + C\dot{w} + Kw = F(t) \tag{33}$$

The damping matrix C is defined as

$$C = \alpha_0 M + \alpha_1 K, \tag{34}$$

and the coefficients α_0 and α_1 are computed by considering damping ratios ξ_1 and ξ_2 in the first two modes of vibration (with corresponding frequencies ω_1 and ω_2) as follows:

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{\omega_1} & \omega_1 \\ \frac{1}{\omega_2} & \omega_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{35}$$

Wilson-θ method is used for the solution of the transient dynamic problem. This method is an implicit integration method and involves computation of dynamic response of a MDOF system by adopting a step by step integration process in the time domain. The Wilson-θ method assumes a linear variation of acceleration over the time interval $[t, t + \theta\delta t]$, where $\theta \geq 1.0$ and δt is a small time step. It has been shown by Wilson that the method becomes unconditionally stable for $\theta \geq 1.38$.

6.1. Optimisation Approach

Uncertainty is considered in the values of Young's modulus and mass density of steel and load. The solution to the resulting interval MDOF system is obtained by an optimisation procedure. This is done by utilising the $fmincon$ function from the optimisation toolbox of MATLAB (The Mathworks, 2011) which seeks to find the minimum of a constrained non-linear multivariate function. The *fmincon* function finds a constrained minimum of a function $f(x)$ of several variables by solving a problem of the form:

$$\{x\} = fmincon(objfun, \{x_0\}, [A], [B], [A_{eq}], \{b_{eq}\}, \{lb\}, \{ub\}) \tag{36}$$

subject to the inequality constraints

$$Ax \leqslant b, \tag{37}$$

equality constraints

$$A_{eq}x_0 = b_{eq} \tag{38}$$

and bounds

$$lb \leq x \leq ub \tag{39}$$

with x_0 being the starting point for search. The last condition (39) defines a set of lower and upper bounds on the design variables $\{x\}$, so that a solution is found in the range $lb \leq x \leq ub$. Wilson-θ method is extended to compute the interval displacement response as a function of time by formulating it as a MATLAB function *objfun* yielding a single output describing the deterministic transient vertical displacement of a given node of the structure. The displacement response is optimised and the bounds for the displacement response are obtained at each time step δt for $0 \leq t \leq t_{max}$. The normalised uncertainties associated with mass and stiffness and load terms are represented by normalised interval parameters $\boldsymbol{p}_1, \boldsymbol{p}_2$ and $\boldsymbol{p}_3$ respectively. These upper and lower bounds of these interval parameters form the vertices of an uncertainty hypercube $\boldsymbol{p}^I$. Any point (p_1, p_2, p_3) inside this bounds $\{lb\}$ and $\{ub\}$ described in equation (39) are defined as

$$lb = \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix} \text{ and } ub = \begin{bmatrix} \overline{p}_1 \\ \overline{p}_2 \\ \overline{p}_3 \end{bmatrix} \tag{40}$$

Equation (33) is recast in interval parametric form as

$$\boldsymbol{p}_1\, M\ddot{w} + Cw + \boldsymbol{p}_2 Kw = \boldsymbol{p}_3 F(t) \tag{41}$$

where parameters $\boldsymbol{p}_i$ are defined as

$$\boldsymbol{p}_i = [\underline{p}_{i,}\overline{p}_i],\ (i = 1, 2, 3). \tag{42}$$

The objective function can be computed at any point p defined by coordinates (p_1, p_2, p_3) within the hypercube $\boldsymbol{p}^I$ that forms the search domain. Thus, using the procedure described above, the deterministic algorithm is translated to an interval algorithm using the global optimisation based approach. In this approach, the lower and upper bounds of interval displacement $\boldsymbol{w}_n$ at a given node n is determined, taking into account that the uncertain parameters p can vary within their intervals $\boldsymbol{p}^I$. This interval $\boldsymbol{w}_n$ of this displacement is determined by a minimisation and a maximisation over the uncertainty interval $\boldsymbol{p}^I$.

$$\boldsymbol{w}_n = \left[\min_{p \in \boldsymbol{p}^I}(\boldsymbol{w}_n), \max_{p \in \boldsymbol{p}^I}(\boldsymbol{w}_n)\right] \tag{43}$$

This is done by computing the displacement $\{w(x, t)\}$ at any location x along the span of the beam at a given time t, using the following deterministic matrix equation,by implementing Wilson-θ approach. :

$$p_1 M\ddot{w} + Cw + p_2 Kw = p_3 F(t). \tag{44}$$

The time history of the displacement response is obtained by computing the minimum and maximum values of the response at each time step. To compute the displacement at a certain time, it has to be computed at all earlier time steps too. However, in order to reduce the computational cost of the optimisations using the local optimisation algorithm *fmincon*, all function evaluations of the objective function are stored in a database

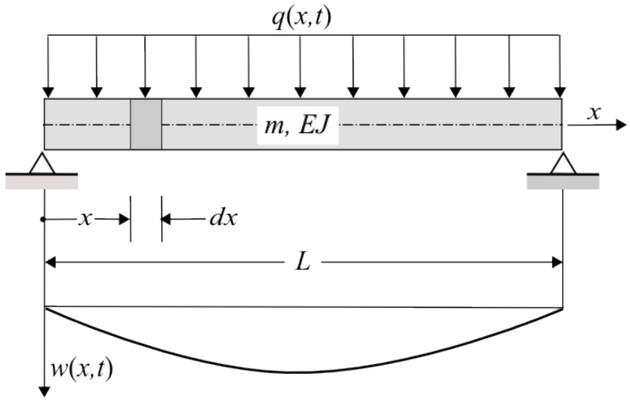


Figure 1. Geometry of the uniform beam with symmetrical load.

to enable the optimiser to reuse them for future optimisations without performing the same finite element analysis again. This database is also used to start optimisations from the point with the best function value found so far the lowest function value for minimisation and the highest function value for maximisation. All optimisations are performed from the highest to the lowest membership level.

7. Numerical experiments

In order to show the interval solutions obtained using the methods described in the paper, an example of the dynamically loaded beam with the load uniformly distributed over the entire span will be considered. Different cases of the amount of uncertainty are investigated.

Example 1. Consider Euler Bernoulli beam shown in Figure 1 with uniform load of 2.5kN applied for a short time of 0.009s. The beam has a span $L = 4$m, area of cross section $A = 0.01m^2$, second moment of area $J = 8.333 \times 10^{-6}\, m^4$ and Young's modulus $E = 200$GPa. It is assumed that mass density is uncertain $\pm 0.5\%$ and the load is uncertain $\pm 20\%$. This gives 2 interval parameters $p = (p_1, p_2) = (\rho, q)$.

The following discretization is applied $n_x = 20$, $(\Delta x = L/n_x)$, number of time steps is equal to $n_t = 100$ and time step is $\Delta t = 0.0015$s. The load is applied for 0.009s.

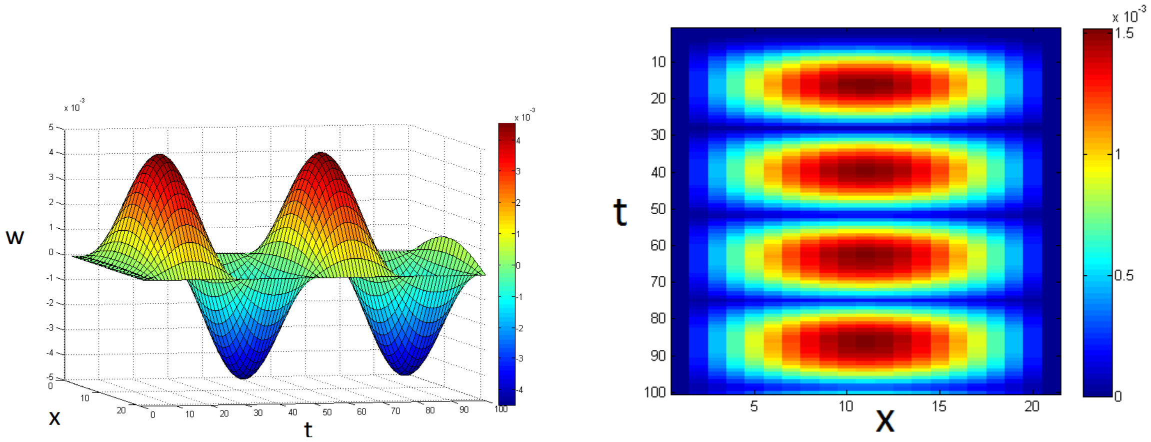
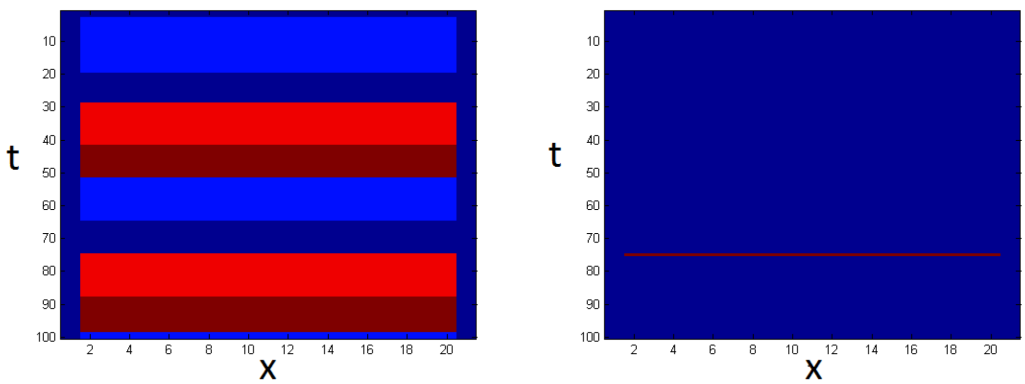


Figure 2. Search method result for $m = 2$, $k = 7$: (a) the lower and upper bound of the interval solution, (b) the difference between upper and lower bound, $\overline{w}^{search} - \underline{w}^{search}$.



(a) (b)

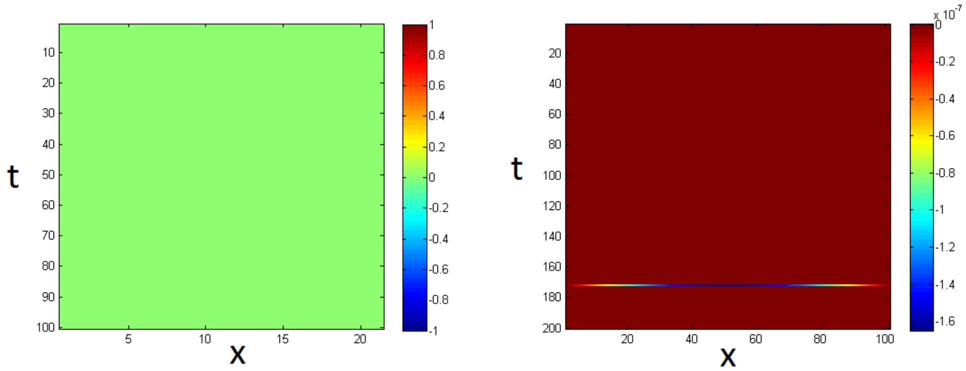
Figure 3. Combinations of parameters which correspond to $p^{\min}$: (a) Search Method, (b) comparison of the Search Method and the Gradient Method. The colours which represent each particular combination for the Search Method and the Gradient Method are different

From the Fig. 3 it is possible to see that the parameters which are calculated by using the Search Method and the Gradient Method are very similar. Only for one time step combinations of parameters were predicted incorrectly by the Gradient Method.

$$p_{i,j}^{\min,gradient} \approx p_{i,j}^{\min,search}. \tag{45}$$

The interval solution depends mostly on the endpoints of the intervals. In order to show this, $p_{i,j}^{\min,search}$, $p_{i,j}^{\max,search}$ is calculated with 2, 5 and 7 intermediate points in the given intervals. For $n_x = 20$, $n_t = 100$ the results are identical:

$$\underline{w}_{i,j}^{search,2} = \underline{w}_{i,j}^{search,5} = \underline{w}_{i,j}^{search,7}. \tag{46}$$



(a) (b)

Figure 4. Difference between the results of the search method $\underline{w}_{i,j}^{search,7} - \underline{w}_{i,j}^{search,2}$ for $k = 2$ and $k = 5$ for different discretization of the problem. $n_x = 20$, $n_t = 100$ (a), $n_x = 100$, $n_t = 200$ (b).

Of course, it is possible to find examples in which there are some difference between the solution for $k = 2$ and $k > 2$. However, according to numerical experiments for the equation which is discussed in this paper $\underline{w}_{i,j}^{search,2} \approx \underline{w}_{i,j}^{search,k}$ and $\overline{w}_{i,j}^{search,2} \approx \overline{w}_{i,j}^{search,k}$ where $k > 2$.

Figure 5 compares the results of the Direct Method and the Search Method. The solution for point data (solid black line) is presented as well.

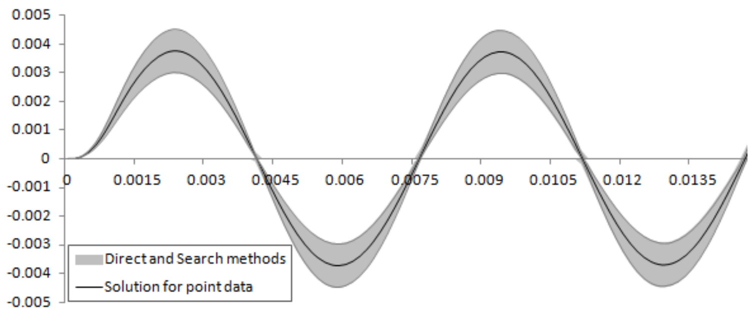


Figure 5. Vertical displacement of the midspan; comparison of the results of Direct method and Search method for the case: $E = 200$[GPa], $\rho = 7850$[kg/m^2]$\pm 0.5\%$, $q = 2.5$[kN]$\pm 20\%$, $n_x = 20$, $n_t = 100$, $\Delta t = 0.0015$.

As can be seen, the results of Direct method and Search method coincide. This proves the quality of the results of both methods.

Example 2. A beam similar to the one used in Example 1 is considered for analysis once again. The beam is acted upon by a load of 5kN/m uniformly distributed over the whole span suddenly for a duration of 0.4 seconds. Five percent damping is considered to be present. The transient dynamic response of the beam is computed using the procedure outlined in section 6.1 and results are presented . Figure 6 shows the time history plot of vertical displacement of the beam at mid-span corresponding to the case with $\boldsymbol{p}_1 = \boldsymbol{p}_2 = [0.95, 1.05]$and $\boldsymbol{p}_3 = [1.0, 1.0]$. This corresponds to deterministic load and interval values of stiffness and mass matrices. Figure 7 shows the time history plot of vertical displacement corresponding to the case $\boldsymbol{p}_1 = \boldsymbol{p}_2 = [1.0, 1.0]$ and $\boldsymbol{p}_3 = [0.8, 1.2]$. Figure 6 clearly depicts the shifting of peaks of response and increase of uncertainty of response as time progresses as uncertainty in mass and stiffness causes a large uncertainty in the eigenfrequencies of the structure. However, no such shifting of peaks is observed in Figure 7 because mass and stiffness properties are deterministic and eigenfrequencies remain deterministic even as time progresses.

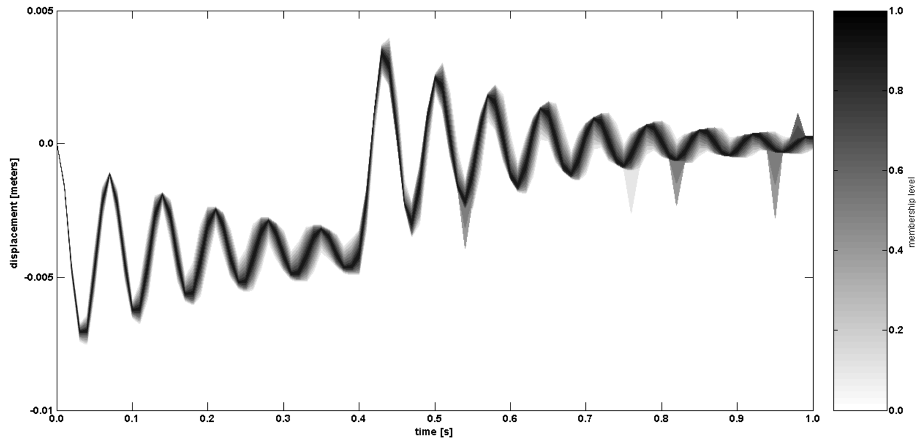


Figure 6. Vertical displacement of the midspan with $\boldsymbol{p}_1 = \boldsymbol{p}_2 = [0.95, 1.05]$, $\mathbf{p}_3 = [1.0, 1.0]$ and 5 percent damping.

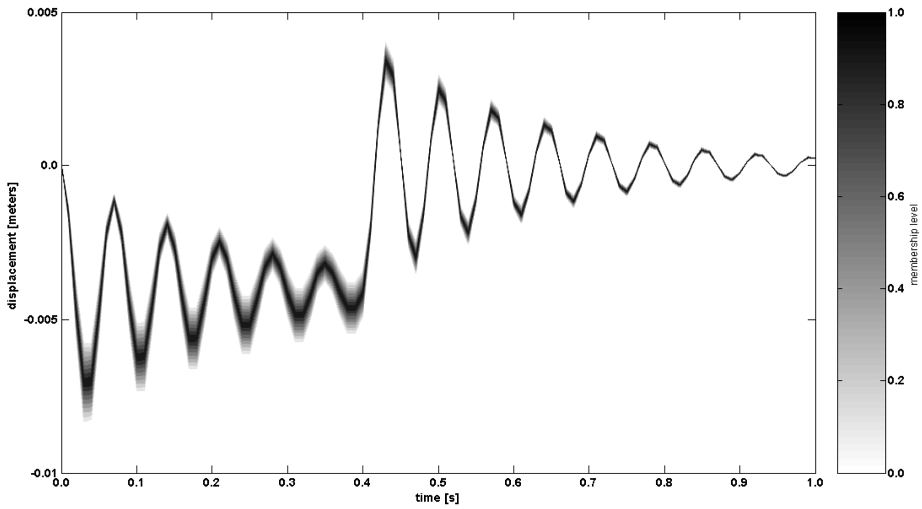


Figure 7. Vertical displacement of midspan with $\boldsymbol{p}_1 = \boldsymbol{p}_2 = [1.0, 1.0], \boldsymbol{p}_3 = [0.8, 1.2]$ and 5 percent damping

8. Conclusions

Several methods for solving beam vibrations problem under interval uncertainty were considered. Based on illustrative examples, it can be stated that in the case of symmetrically loaded beam, the interval solution depends on very few combinations of uncertain parameters. In the example, the interval solution depends only on 4 combinations of parameters. This means that in practice it is possible to find four point solutions in order to compute the exact values of the interval solution. Appropriate combinations of parameters can be predicted by using the gradient of the solution. Moreover, it is possible to increase the accuracy of the calculations by using adaptive approximation (Pownuk, 2011) which will be a topic of future research. In more complex cases, as seen in Example 1, it is possible to find large areas in which the interval solution depends only on specific combinations of parameters. It is possible to use this information in order to improve accuracy of the interval solution. There are also situations in which the interval solution depends on infinite number of combinations of the parameters. According to numerical results, in considered example, the solution depends only on the endpoints of the parameters. In such situations it is possible to calculate the exact solution by using the gradient method (Pownuk, 2004). In this case it is possible to calculate the interval solution approximately using Taylor method, which is especially useful for narrow intervals. The guaranteed solution can be obtained using the Direct method for solving parametric interval linear systems, however the method requires some improvement do deal with large scale problems. Direct formulation of the iterative problems can eliminate the wrapping effects from the interval calculations. Optimisation approach is time consuming but produces acceptable results even with large intervals of input parameters.

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