Frequency response functions of discretized structural systems with uncertain parameters

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Abstract: A procedure for deriving in explicit approximate form the *frequency response function* (*FRF*) of linear discretized structures with uncertain stiffness properties is presented. The proposed procedure is based on the following main steps: *i*) to perform the spectral decomposition of the deviation of the stiffness matrix (with respect to its nominal value) so as to obtain a sum of rank-one matrices, each one associated to a single uncertain parameter; *ii*) to project the equations of motion in the modal subspace; *iii*) to introduce a novel series expansion of the *FRF* in the modal subspace which provides an approximate, but explicit, expression of the *FRF* of structural systems with uncertain parameters. The potential of the proposed series expansion are demonstrated in the context of the so-called *improved interval analysis* by determining the range of the modulus of the *FRF* of structures with uncertain-but-bounded parameters.

Keywords: Frequency response function; Uncertain parameters; Spectral decomposition; Improved interval analysis.

1. Introduction

In Structural Dynamics, the *frequency response function* (*FRF*), also called *transfer function*, is a complex function able to provide information about the behavior of a structure over a range of frequencies. For instance, the frequency domain response of a single-degree-of-freedom system (SDOF), i.e. an oscillator, is evaluated simply multiplying the *FRF* by the Fourier transform of the forcing function. For multi-DOF structural systems the *FRF* describes the relationship between a local excitation applied at one location on the structure and the resulting response at another and/or the same location. The frequency domain approach often gives information useful for structural design purposes that cannot be alternatively caught by the time domain approach. Moreover, it is sometimes more convenient to perform the analysis in the frequency domain; as an example, for structures with frequency dependent parameters or subjected to stationary random processes and so on. Indeed, in all these cases the evaluation of the *FRF* is required.

In practical engineering problems, material properties, geometry and boundary conditions of a structure may experience fluctuations, due to measurement and manufacturing errors or other factors, which may significantly affect the response. The uncertainties are usually described following two contrasting points of view, known as *probabilistic* and *non-probabilistic approaches*. The probabilistic approach requires a wealth of data, often unavailable, to define the probability distribution density of the uncertain structural

5th International Conference on Reliable Engineering Computing (REC 2012)
Edited by M. Vořechovský, V. Sadílek, S. Seitl, V. Veselý, R. L. Muhanna and R. L. Mullen

ISBN 978-80-214-4507-9. Published by Ing. Vladislav Pokorný − LITERA

parameters. If available information is fragmentary or incomplete, non-probabilistic approaches, such as convex models, fuzzy set theory or interval models (Elishakoff and Ohsaki, 2010), can be alternatively applied.

Among non-probabilistic approaches, the *interval model* turns out to be the most suitable approach when only the upper and lower bounds of a non-deterministic property are well defined. Indeed, this model is derived from the interval analysis (Moore, 1966; Alefeld and Herzberger, 1983; Moore et al., 2009) in which the number is treated as an interval variable ranging between its lower and upper bounds. Unfortunately, the "ordinary" interval analysis (Moore, 1966) suffers from the so-called *dependency phenomenon* (Muhanna and Mullen, 2001; Moens and Vandepitte, 2005; Moore et al., 2009) which often leads to an overestimation of the interval width that could be catastrophic from an engineering point of view. This occurs when an expression contains multiple instances of one or more interval variables. Indeed, the ordinary interval arithmetic operations erroneously assume that the operand interval numbers are independent. To limit the catastrophic effects of the dependency phenomenon, the so-called *generalized interval analysis* (Hansen, 1975) and the *affine arithmetic* (Comba and Stolfi, 1993; Stolfi and De Figueiredo, 2003) have been introduced in the literature. In these formulations, each intermediate result is represented by a linear function with a small remainder interval (Nedialkov et al., 2004).

In the framework of probabilistic approaches, the *FRF* has been evaluated by Falsone and Ferro (2005, 2007) in explicit form by taking into account the properties of the natural deformation modes of the finite element discretized structure. In a non-probabilistic context, Moens and Vandepitte (2004) proposed a numerical procedure to efficiently calculate close outer approximations on the envelope *FRF* of structures with interval uncertainties. The *FRF* of systems with uncertain-but-bounded parameters was also evaluated by Manson (2005) employing both the *complex interval analysis* and the *complex affine arithmetic*.

In this paper, an alternative approach for the evaluation of the *FRF* of discretized structures with uncertain stiffness properties is presented. The proposed procedure requires the following preliminary steps: *i*) the spectral decomposition of the deviation of the stiffness matrix (with respect to its nominal value) to obtain a sum of rank-one matrices, each one associated to a single uncertain parameter; *ii*) the modal analysis to project the equations of motion in the modal subspace. In a second stage, a novel series expansion of the modal *FRF*, named *Rational Series Expansion* (*RSE*), which provides an approximate, but explicit, expression of the *FRF* of structural systems with uncertain parameters, is derived. Finally, the proposed series expansion together with the so-called *improved interval analysis* presented by Muscolino and Sofi (2011) is used to obtain the range of the modulus of the *FRFs* of structures with uncertain-but-bounded parameters.

Numerical applications performed on a truss structure and a portal frame with uncertain Young's moduli of the material have demonstrated the accuracy of the proposed explicit approximation of the *FRF*.

2. Preliminary concepts

2.1. Equations of motion

Let us consider a quiescent n-DOF linear structural system with uncertain stiffness properties subjected to the forcing vector $\mathbf{f}(t)$. The equations of motion can be cast in the form:

$$\mathbf{M}\ddot{\mathbf{u}}(\boldsymbol{\alpha},t)+\mathbf{C}\dot{\mathbf{u}}(\boldsymbol{\alpha},t)+\mathbf{K}(\boldsymbol{\alpha})\mathbf{u}(\boldsymbol{\alpha},t)=\mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$ and $\mathbf{C}$ are the $n\times n$ mass and damping matrices of the structure; $\boldsymbol{\alpha}=[\alpha_1,\ \alpha_2,...,\ \alpha_r]^{\mathrm{T}}$ is the vector collecting the r dimensionless uncertain parameters α_i; $\mathbf{u}(\boldsymbol{\alpha},t)$ is the vector of nodal displacements and a dot over a variable denotes differentiation with respect to time t.

It is worth noting that the relationship between the stiffness matrix, $\mathbf{K}(\boldsymbol{\alpha})$, and the vector $\boldsymbol{\alpha}$ is often linear or, by applying a suitable variable transformation, it is always possible to make the stiffness matrix depend linearly on the new variables. Based on this concept, the stiffness matrix $\mathbf{K}(\boldsymbol{\alpha})$ is herein expressed as a linear function of the uncertain properties, i.e.:

$$\mathbf{K}(\boldsymbol{\alpha})=\mathbf{K}_0+\Delta\mathbf{K}(\boldsymbol{\alpha})=\mathbf{K}_0+\sum_{i=1}^{r}\mathbf{K}_i\,\alpha_i\,; \tag{2}$$

$$\mathbf{K}_0=\mathbf{K}(\boldsymbol{\alpha}_0);\qquad \mathbf{K}_i=\frac{\partial}{\partial\alpha_i}\mathbf{K}(\boldsymbol{\alpha})\bigg|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \tag{3}$$

where $\mathbf{K}_0$ is the nominal value of the stiffness matrix, which is a positive definite symmetric matrix of order $n\times n$, $\mathbf{K}_i$ is a semi-positive definite symmetric matrix of order $n\times n$ and rank p_i and α_i is the i-th dimensionless uncertain parameter. In structural engineering problems, the fluctuating properties can be reasonably assumed to satisfy the condition $|\alpha_i|<1$, with the symbol $|\bullet|$ denoting absolute value.

In the framework of the traditional modal analysis, the solution of the equations of motion (1) may be pursued by introducing the following coordinate transformation:

$$\mathbf{u}(\boldsymbol{\alpha},t)=\boldsymbol{\Phi}_0\,\mathbf{q}(\boldsymbol{\alpha},t) \tag{4}$$

where $\mathbf{q}(\boldsymbol{\alpha},t)$ is the vector gathering the first m modal coordinates $q_j(\boldsymbol{\alpha},t)$ ($j=1,2,...,m\leq n$); $\boldsymbol{\Phi}_0$ is the modal matrix, of order $n\times m$, pertaining to the nominal configuration in which $\mathbf{K}_0=\mathbf{K}(\boldsymbol{\alpha}_0)$. Specifically, the modal matrix $\boldsymbol{\Phi}_0$, collecting the first m eigenvectors normalized with respect to the mass matrix $\mathbf{M}$, is evaluated as solution of the following eigenproblem:

$$\mathbf{K}_0\boldsymbol{\Phi}_0=\mathbf{M}\boldsymbol{\Phi}_0\,\boldsymbol{\Omega}_0^2\,;\qquad \boldsymbol{\Phi}_0^{\mathrm{T}}\mathbf{M}\boldsymbol{\Phi}_0=\mathbf{I}_m \tag{5}$$

where $\mathbf{I}_m$ denotes the identity matrix of order m; $\boldsymbol{\Omega}_0^2=\boldsymbol{\Phi}_0^{\mathrm{T}}\mathbf{K}_0\boldsymbol{\Phi}_0$ is the spectral matrix of the nominal structural system, say a diagonal matrix listing the squares of the natural circular frequencies of the structure, $\omega_{0,i}$, for the nominal values of the uncertain parameters; the apex T means transpose matrix. By applying the coordinate transformation (4), the equations of motion (1) can be projected in the modal space:

$$\ddot{\mathbf{q}}(\boldsymbol{\alpha},t)+\boldsymbol{\Xi}\,\dot{\mathbf{q}}(\boldsymbol{\alpha},t)+\boldsymbol{\Omega}^2(\boldsymbol{\alpha})\,\mathbf{q}(\boldsymbol{\alpha},t)=\mathbf{p}(t) \tag{6}$$

where $\boldsymbol{\Omega}^2(\boldsymbol{\alpha})=\boldsymbol{\Phi}_0^{\mathrm{T}}\mathbf{K}(\boldsymbol{\alpha})\boldsymbol{\Phi}_0$; $\boldsymbol{\Xi}=\boldsymbol{\Phi}_0^{\mathrm{T}}\mathbf{C}\boldsymbol{\Phi}_0$ is the generalised damping matrix, which for classically damped systems is a diagonal one; $\mathbf{p}(t)=\boldsymbol{\Phi}_0^{\mathrm{T}}\mathbf{f}(t)$ is the modal forcing vector. Notice that by virtue of the decomposition (2) of the stiffness matrix, the following relationship holds:

$$\mathbf{\Omega}^2(\boldsymbol{\alpha}) = \mathbf{\Phi}_0^{\mathrm{T}}\mathbf{K}(\boldsymbol{\alpha})\mathbf{\Phi}_0 = \mathbf{\Phi}_0^{\mathrm{T}}\mathbf{K}_0\mathbf{\Phi}_0 + \sum_{i=1}^{r}\mathbf{\Phi}_0^{\mathrm{T}}\mathbf{K}_i\mathbf{\Phi}_0\alpha_i = \mathbf{\Omega}_0^2 + \sum_{i=1}^{r}\mathbf{\Omega}_i^2\alpha_i \tag{7}$$

where

$$\mathbf{\Omega}_i^2 = \mathbf{\Phi}_0^{\mathrm{T}}\mathbf{K}_i\mathbf{\Phi}_0 \tag{8}$$

is not a diagonal matrix.

2.2. Frequency domain response

In some cases, such as for structures with frequency dependent parameters or in presence of stochastic stationary excitations, it is more convenient to perform the analysis in the so-called frequency domain.

In the context of the frequency domain analysis, it is assumed that the loading is periodic and has been resolved into its discrete harmonic components by Fourier transformation. The corresponding harmonic components of the structural response can be derived by performing the Fourier transform of both sides of Eq. (6) (or Eq.(1)) obtaining the following set of algebraic frequency dependent equations:

$$\left[-\omega^2\,\mathbf{I}_m + \mathrm{i}\,\omega\,\mathbf{\Xi} + \mathbf{\Omega}^2(\boldsymbol{\alpha})\right]\mathbf{Q}(\boldsymbol{\alpha},\omega) = \mathbf{P}(\omega) \tag{9}$$

where $\mathbf{Q}(\boldsymbol{\alpha},\omega)$ and $\mathbf{P}(\omega)$ are the vectors collecting the Fourier transforms of $\mathbf{q}(\boldsymbol{\alpha},t)$ and $\mathbf{p}(t)$, respectively. The modal frequency response vector $\mathbf{Q}(\boldsymbol{\alpha},\omega)$, solution of Eq.(9), can be expressed as follows:

$$\mathbf{Q}(\boldsymbol{\alpha},\omega) = \mathbf{H}(\boldsymbol{\alpha},\omega)\mathbf{P}(\omega) \tag{10}$$

where

$$\mathbf{H}(\boldsymbol{\alpha},\omega) = \left[-\omega^2\,\mathbf{I}_m + \mathrm{i}\,\omega\,\mathbf{\Xi} + \mathbf{\Omega}^2(\boldsymbol{\alpha})\right]^{-1} = \left[\mathbf{H}_0^{-1}(\omega) + \sum_{i=1}^{r}\mathbf{\Omega}_i^2\,\alpha_i\right]^{-1} \tag{11}$$

is the *modal frequency response function (FRF) matrix* (referred to also as *transfer function matrix*) whose expression has been derived taking into account Eq. (7) and introducing the *FRF* matrix of the nominal structural system, given by:

$$\mathbf{H}_0(\omega) = \left[-\omega^2\,\mathbf{I}_m + \mathrm{i}\,\omega\,\mathbf{\Xi} + \mathbf{\Omega}_0^2\right]^{-1}. \tag{12}$$

It is worth noting that the *FRF* matrix $\mathbf{H}(\boldsymbol{\alpha},\omega)$ is not diagonal, while for classically damped systems the matrix $\mathbf{H}_0(\omega)$ is a diagonal one.

Once the modal frequency response $\mathbf{Q}(\boldsymbol{\alpha},\omega)$ is evaluated, the frequency response $\mathbf{U}(\boldsymbol{\alpha},\omega)$ in the nodal space can be obtained by performing the Fourier Transform of Eq.(4), i.e.:

$$\mathbf{U}(\boldsymbol{\alpha},\omega) = \mathbf{\Phi}_0\,\mathbf{Q}(\boldsymbol{\alpha},\omega). \tag{13}$$

To avoid the inversion of the parametric frequency dependent matrix in Eq.(11), the Neumann series expansion can be adopted which leads to the following expression:

$$\mathbf{H}(\boldsymbol{\alpha},\omega)=\left[\mathbf{I}_m+\mathbf{H}_0(\omega)\sum_{i=1}^{r}\boldsymbol{\Omega}_i^2\alpha_i\right]^{-1}\mathbf{H}_0(\omega)=\mathbf{H}_0(\omega)+\sum_{k=1}^{\infty}\left(-1\right)^k\left[\mathbf{H}_0(\omega)\sum_{i=1}^{r}\boldsymbol{\Omega}_i^2\alpha_i\right]^k\mathbf{H}_0(\omega). \quad (14)$$

The convergence of this series expansion is guaranteed if and only if the least square norm of the matrix in square brackets is less than one. In the next section, an alternative series expansion of the modal *FRF* matrix for structural systems with uncertain parameters is proposed.

3. Proposed explicit form of the FRF matrix

3.1. Spectral decomposition of the stiffness matrix

As well known, in structural engineering the stiffness matrix is always a positive definite matrix. In the previous section, the stiffness matrix has been assumed to depend on r dimensionless uncertain parameters satisfying the conditions $|\alpha_i| < 1$, i.e. $\mathbf{K} = \mathbf{K}(\boldsymbol{\alpha})$. Furthermore, the stiffness matrix has been decomposed according to Eq.(2), where $\mathbf{K}_0$ is a positive definite symmetric matrix of order $n\times n$, while $\mathbf{K}_i$ is a semi-positive symmetric matrix of order $n\times n$ and rank p_i. As an example, in the case of truss structures and shear-type frames, the matrices $\mathbf{K}_i$ have rank $p_i = 1$. Instead, for flexible frames the matrix $\mathbf{K}_i$ has rank $p_i = 3$ and so on. The foregoing property can be exploited to perform the *spectral decomposition* (referred to also as *eigendecomposition*) of the matrices $\mathbf{K}_i$. To this aim, the following eigenproblems have to be solved:

$$\mathbf{K}_i\boldsymbol{\psi}_i^{(\ell)}=\lambda_i^{(\ell)}\mathbf{K}_0\boldsymbol{\psi}_i^{(\ell)}, \quad (i=1,2,...,r;\ \ell=1,2,...,p_i) \quad (15)$$

where $\lambda_i^{(\ell)}$ denote the eigenvalues which are real positive numbers, while $\boldsymbol{\psi}_i^{(\ell)}$ are the associated eigenvectors. Due to the semi-positivity of the matrix $\mathbf{K}_i$, among the n eigenvalues of the i-th eigenproblem in Eq. (15) only $p_i < n$ eigenvalues are different from zero. As an example, in the case of truss structures and shear-type frames only one eigenvalue different from zero is found for each uncertain parameter; for flexible frames, each eigenproblem (15) yields three eigenvalues different from zero and so on.

By imposing that the eigenvectors $\boldsymbol{\psi}_i^{(\ell)}$ satisfy the orthonormalization condition:

$$\boldsymbol{\Psi}_i^{\mathrm{T}}\mathbf{K}_0\boldsymbol{\Psi}_i=\mathbf{I}_{p_i};\quad \boldsymbol{\Psi}_i=\left[\boldsymbol{\psi}_i^{(1)}\quad \boldsymbol{\psi}_i^{(2)}\quad \cdots\quad \boldsymbol{\psi}_i^{(p_i)}\right], \quad (16)$$

the following relationship holds:

$$\boldsymbol{\Psi}_i^{\mathrm{T}}\mathbf{K}_i\boldsymbol{\Psi}_i=\boldsymbol{\Lambda}_i;\quad \boldsymbol{\Lambda}_i=\mathrm{Diag}\left[\lambda_i^{(1)},\quad \lambda_i^{(2)}\quad ,\cdots,\quad \lambda_i^{(p_i)}\right]. \quad (17)$$

Then, after very simple algebra, by applying the previously described spectral decomposition, the matrix $\mathbf{K}_i$ can be written as:

$$\mathbf{K}_i = \left(\mathbf{K}_0 \mathbf{\Psi}_i\right) \mathbf{\Lambda}_i \left(\mathbf{\Psi}_i^{\mathrm{T}} \mathbf{K}_0\right) = \sum_{\ell=1}^{p_i} \lambda_i^{(\ell)} \left(\mathbf{K}_0 \boldsymbol{\psi}_i^{(\ell)}\right)\left(\boldsymbol{\psi}_i^{(\ell)\mathrm{T}} \mathbf{K}_0\right) = \sum_{\ell=1}^{p_i} \lambda_i^{(\ell)} \mathbf{v}_i^{(\ell)} \, \mathbf{v}_i^{(\ell)\mathrm{T}} \tag{18}$$

where

$$\mathbf{v}_i^{(\ell)} = \mathbf{K}_0 \, \boldsymbol{\psi}_i^{(\ell)}. \tag{19}$$

Substituting Eq.(18) into Eq. (2), the stiffness matrix $\mathbf{K}(\boldsymbol{\alpha})$ can be expressed as the superposition of $p = \sum_{i=1}^{r} p_i$ changes of rank-one, i.e.:

$$\mathbf{K}(\boldsymbol{\alpha}) = \mathbf{K}_0 + \Delta\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{r} \alpha_i \left[\sum_{\ell=1}^{p_i} \lambda_i^{(\ell)} \mathbf{v}_i^{(\ell)} \, \mathbf{v}_i^{(\ell)\mathrm{T}} \right]. \tag{20}$$

Finally, upon introducing the spectral decomposition of the stiffness matrix given by Eq.(20) into Eq. (7), the matrix $\mathbf{\Omega}^2(\boldsymbol{\alpha})$, appearing in the *FRF* matrix (11), takes the following form:

$$\mathbf{\Omega}^2(\boldsymbol{\alpha}) = \mathbf{\Phi}_0^{\mathrm{T}} \mathbf{K}(\boldsymbol{\alpha}) \mathbf{\Phi}_0 = \mathbf{\Omega}_0^2 + \sum_{i=1}^{r} \mathbf{\Omega}_i^2 \alpha_i = \mathbf{\Omega}_0^2 + \sum_{i=1}^{r} \alpha_i \left[\sum_{\ell=1}^{p_i} \lambda_i^{(\ell)} \mathbf{w}_i^{(\ell)} \, \mathbf{w}_i^{(\ell)\mathrm{T}} \right] \tag{21}$$

where

$$\mathbf{\Omega}_i^2 = \sum_{\ell=1}^{p_i} \lambda_i^{(\ell)} \mathbf{w}_i^{(\ell)} \, \mathbf{w}_i^{(\ell)\mathrm{T}} \tag{22}$$

with

$$\mathbf{w}_i^{(\ell)} = \mathbf{\Phi}_0^{\mathrm{T}} \mathbf{K}_0 \, \boldsymbol{\psi}_i^{(\ell)}. \tag{23}$$

3.2. APPROXIMATE MODAL FRF MATRIX FOR TRUSS STRUCTURES WITH UNCERTAIN PARAMETERS

In order to illustrate the proposed procedure for the derivation of an explicit approximate form of the *FRF* matrix, the simplest case of truss structures is first examined. In particular, recalling that for truss structures the *i*-th eigenproblem in Eq. (15) gives only one eigenvalue different from zero, i.e. $p_i = 1$, $\lambda_I = \lambda_i^{(1)}$ and $\boldsymbol{\psi}_i = \boldsymbol{\psi}_i^{(1)}$, the spectral decomposition of the matrix $\mathbf{K}_i$ outlined in the previous section reduces to:

$$\mathbf{K}_i = \lambda_i \, \mathbf{v}_i \, \mathbf{v}_i^{\mathrm{T}} \tag{24}$$

where $\mathbf{v}_i = \mathbf{v}_i^{(1)}$. Accordingly, the matrix $\mathbf{\Omega}_i^2(\boldsymbol{\alpha})$ in Eq. (22) takes the following form:

$$\mathbf{\Omega}_i^2 = \alpha_i \lambda_i \, \mathbf{w}_i \, \mathbf{w}_i^{\mathrm{T}} \tag{25}$$

where $\mathbf{w}_i = \mathbf{w}_i^{(1)}$. By substituting Eq. (25) into Eq. (14), the Neumann series expansion of the modal *FRF* matrix can be rewritten as:

$$\mathbf{H}(\boldsymbol{\alpha},\omega) = \mathbf{H}_0(\omega) + \sum_{s=1}^{\infty} (-1)^s \left[\mathbf{H}_0(\omega) \sum_{i=1}^{r} \alpha_i \, \lambda_i \, \mathbf{w}_i \, \mathbf{w}_i^{\mathrm{T}} \right]^s \mathbf{H}_0(\omega). \tag{26}$$

In order to improve the convergence, the terms into square brackets in Eq.(26) are herein rewritten in explicit form obtaining the following expression of the *FRF* matrix, named *Rational Series Expansion* (*RSE*):

$$\begin{aligned}
\mathbf{H}(\boldsymbol{\alpha},\omega) = \left[\mathbf{H}_0^{-1}(\omega) + \sum_{i=1}^{r} \alpha_i \, \lambda_i \, \mathbf{w}_i \, \mathbf{w}_i^{\mathrm{T}} \right]^{-1} &\approx \mathbf{H}_0(\omega) - \sum_{i=1}^{r} \frac{\alpha_i \lambda_i}{1+\alpha_i \lambda_i \, d_i\,(\omega)} \, \mathbf{D}_i(\omega) \\
&+ \sum_{i=1}^{r} \sum_{j=i+1}^{r} \frac{\alpha_i \alpha_j \, \lambda_j \, \lambda_i}{1+\alpha_j \lambda_j d_{ij}(\omega)} d_{ij}(\omega) \left[\mathbf{D}_{ij}(\omega) + \mathbf{D}_{ij}{}^{\mathrm{T}}(\omega) \right] - \\
&- \sum_{i=1}^{r} \sum_{\substack{j=1 \\ j \neq i}}^{r} \sum_{\substack{k=1 \\ k \neq j}}^{r} \frac{\alpha_i \alpha_j \alpha_k \, \lambda_i \, \lambda_j \lambda_k}{1+\alpha_k \lambda_k d_{jk}(\omega)} \, d_{ij}(\omega) d_{jk}(\omega) \mathbf{D}_{ik}(\omega) + \\
&+ \sum_{i=1}^{r} \sum_{\substack{j=1 \\ j \neq i}}^{r} \sum_{\substack{k=1 \\ k \neq j}}^{r} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{r} \frac{\alpha_i \alpha_j \alpha_k \alpha_\ell \lambda_i \, \lambda_j \lambda_k \lambda_\ell}{1+\alpha_\ell \lambda_\ell d_{k\ell}(\omega)} \, d_{ij}(\omega) d_{jk}(\omega) d_{k\ell}(\omega) \mathbf{D}_{i\ell}(\omega) - \cdots
\end{aligned} \tag{27}$$

where only the first four terms have been retained and the following complex quantities have been introduced:

$$d_i(\omega) = \mathbf{w}_i^{\mathrm{T}} \mathbf{H}_0(\omega) \mathbf{w}_i \, ; \qquad \mathbf{D}_i(\omega) = \mathbf{H}_0(\omega) \mathbf{w}_i \mathbf{w}_i^{\mathrm{T}} \mathbf{H}_0(\omega); \tag{28}$$

$$d_{is}(\omega) = \mathbf{w}_i^{\mathrm{T}} \mathbf{H}_0(\omega) \mathbf{w}_s \, ; \qquad \mathbf{D}_{is}(\omega) = \mathbf{H}_0(\omega) \mathbf{w}_i \mathbf{w}_s^{\mathrm{T}} \mathbf{H}_0(\omega); \quad (s = j, k, \ell, m, \ldots). \tag{29}$$

Equation (27) holds if and only if the following conditions are satisfied:

$$\left\| \alpha_i \lambda_i d_i(\omega) \right\| < 1; \qquad \left\| \alpha_j \lambda_j d_{ij}(\omega) \right\| < 1; \ldots \tag{30}$$

where the symbol $\|\bullet\|$ denotes the modulus of $\bullet$.

Moreover, if $|\alpha_s| << 1$, the approximate modal *FRF* matrix can be accurately evaluated by retaining only first-order terms of the *RSE* in Eq.(27), i.e.:

$$\mathbf{H}(\boldsymbol{\alpha},\omega)=\left[\mathbf{H}_0^{-1}(\omega)+\sum_{i=1}^{r}\alpha_i\,\lambda_i\,\mathbf{w}_i\,\mathbf{w}_i^{\mathrm{T}}\right]^{-1}\approx\mathbf{H}_0(\omega)-\sum_{i=1}^{r}\frac{\alpha_i\lambda_i}{1+\alpha_i\lambda_i\,d_i\,(\omega)}\,\mathbf{D}_i(\omega). \tag{31}$$

It has to be emphasized that Eqs. (27) and (31) provide with different levels of accuracy closed form expressions of the modal *FRF* matrix of truss structures with uncertain parameters. This remarkable result can be exploited to derive explicit solutions for the frequency domain response of truss structures with fluctuating parameters.

3.3. APPROXIMATE MODAL FRF MATRIX FOR THE MOST GENERAL CASE OF DISCRETIZED STRUCTURES

In this section, an approximate closed form expression of the *FRF* matrix for the most general case of discretized structural systems is derived by applying the procedure described above for truss structures. Specifically, taking into account that in this case the *i*-th eigenproblem in Eq. (15) gives p_i eigenvalues different from zero, the spectral decomposition of the stiffness matrix leads to Eq. (22) for the matrix $\boldsymbol{\Omega}_i^2$. Substituting this expression into Eq.(14) and rewriting the terms of the Neumann series expansion according to Eq.(27), the modal *FRF* matrix can be approximated in explicit form by the following *RSE*:

$$\begin{aligned}\mathbf{H}(\boldsymbol{\alpha},\omega)\approx{}&\mathbf{H}_0(\omega)-\sum_{i=1}^{r}\sum_{\ell=1}^{p_i}\frac{\alpha_i\,\lambda_i^{(\ell)}}{1+\alpha_i\,\lambda_i^{(\ell)}\,b_{i\ell}(\omega)}\;\mathbf{B}_{i\ell}\,(\omega)+\\&+\sum_{i=1}^{r}\sum_{\ell=1}^{p_i}\sum_{\substack{j=1\\j\neq i}}^{r}\sum_{m=1}^{p_j}\frac{\alpha_i\alpha_j\,\lambda_i^{(\ell)}\lambda_j^{(m)}}{1+\alpha_j\lambda_j^{(m)}\,b_{ij\ell m}(\omega)}\,b_{ij\ell m}(\omega)\mathbf{B}_{ij\ell m}(\omega)+\\&-\sum_{i=1}^{r}\sum_{\ell=1}^{p_i}\sum_{\substack{j=1\\j\neq i}}^{r}\sum_{m=1}^{p_j}\sum_{\substack{k=1\\k\neq j}}^{r}\sum_{n=1}^{p_k}\frac{\alpha_i\alpha_j\alpha_k\lambda_i^{(\ell)}\lambda_j^{(m)}\lambda_k^{(n)}}{1+\alpha_k\lambda_k^{(n)}\,b_{jkmn}(\omega)}\,b_{jkmn}(\omega)b_{ij\ell m}(\omega)\mathbf{B}_{ik\ell n}(\omega)+\cdots\end{aligned} \tag{32}$$

where

$$b_{i\ell}(\omega)=\mathbf{w}_i^{(\ell)\mathrm{T}}\mathbf{H}_0(\omega)\,\mathbf{w}_i^{(\ell)};\qquad \mathbf{B}_{i\ell}(\omega)=\mathbf{H}_0(\omega)\,\mathbf{w}_i^{(\ell)}\;\mathbf{w}_i^{(\ell)\mathrm{T}}\mathbf{H}_0(\omega); \tag{33}$$

$$b_{ij\ell m}(\omega)=\mathbf{w}_i^{(\ell)\mathrm{T}}\mathbf{H}_0(\omega)\,\mathbf{w}_j^{(m)}\;;\qquad \mathbf{B}_{ij\ell m}(\omega)=\mathbf{H}_0(\omega)\,\mathbf{w}_i^{(\ell)}\;\mathbf{w}_j^{(m)\mathrm{T}}\mathbf{H}_0(\omega); \tag{34}$$

$$b_{jkmn}(\omega)=\mathbf{w}_j^{(m)\mathrm{T}}\mathbf{H}_0(\omega)\,\mathbf{w}_k^{(n)}\;;\qquad \mathbf{B}_{ik\ell n}(\omega)=\mathbf{H}_0(\omega)\,\mathbf{w}_i^{(\ell)}\;\mathbf{w}_k^{(n)\mathrm{T}}\mathbf{H}_0(\omega) \tag{35}$$

are complex quantities. Obviously, Eq.(32) holds if and only if the following conditions are satisfied:

$$\left\|\alpha_i\lambda_i^{(\ell)}\,b_{i\ell}(\omega)\right\|<1;\quad\left\|\alpha_j\lambda_j^{(m)}\,b_{ij\ell m}(\omega)\right\|<1;\quad\left\|\alpha_k\lambda_k^{(n)}\,b_{jkmn}(\omega)\right\|<1;\ldots \tag{36}$$

If the uncertain parameters satisfy the condition $|\alpha_s|<<1$, an accurate approximation of the modal *FRF* matrix can be obtained by retaining only first-order terms of the *RSE* in Eq.(32), i.e.:

$$\mathbf{H}(\boldsymbol{\alpha},\omega)=\left[\mathbf{H}_0^{-1}(\omega)+\sum_{i=1}^{r}\alpha_i\left(\sum_{\ell=1}^{p_i}\lambda_i^{(\ell)}\mathbf{w}_i^{(\ell)}\,\mathbf{w}_i^{(\ell)\mathrm{T}}\right)\right]^{-1}\approx\mathbf{H}_0(\omega)-\sum_{i=1}^{r}\sum_{\ell=1}^{p_i}\frac{\alpha_i\,\lambda_i^{(\ell)}}{1+\alpha_i\,\lambda_i^{(\ell)}\,b_{i\ell}(\omega)}\,\mathbf{B}_{i\ell}\,(\omega). \quad (37)$$

Equations (32) and (37) represent closed form expressions which approximate with different accuracy the modal *FRF* matrix of discretized structures with uncertain parameters. Such expressions are very useful to investigate the effects of the fluctuating properties on the frequency domain response of discretized structures, since the response can be derived in explicit form as well.

4. Uncertain-but-bounded parameters

4.1. PRELIMINARY DEFINITIONS: REAL AND COMPLEX INTERVAL VARIABLES

In this section, the *r* uncertain structural parameters α_i ($i = 1,2,\dots,r$) introduced in the above formulation are assumed independent and are modeled as interval variables. Then, according to the *"ordinary" interval analysis* (Moore, 1966; Alefeld and Herzberger, 1983; Neumaier, 1990; Moore et al., 2009), denoting by $\mathbb{IR}$ the set of all closed real interval numbers, the bounded set-interval vector of real numbers $\boldsymbol{\alpha}^I \triangleq [\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}]\in\mathbb{IR}^r$, such that $\underline{\boldsymbol{\alpha}}\le\boldsymbol{\alpha}\le\overline{\boldsymbol{\alpha}}$, can be introduced. The apex *I* characterizes the interval variables, while $\underline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\alpha}}$ denote the vectors collecting the lower and upper bounds of the *i*-th uncertain parameter $\alpha_i^I\in\mathbb{IR}$, say $\underline{\alpha}_i$ and $\overline{\alpha}_i$.

Unfortunately, the "ordinary" interval analysis suffers from the so-called *dependency phenomenon* (Muhanna and Mullen, 2001; Moens and Vandepitte, 2005; Moore et al., 2009) which often leads to an overestimation of the interval width that could be catastrophic from an engineering point of view. This occurs when an expression contains multiple instances of one or more interval variables. To limit the catastrophic effects of the dependency phenomenon, the so-called *generalized interval analysis* (Hansen, 1975) and *affine arithmetic* (Comba and Stolfi, 1993; Stolfi and De Figueiredo, 2003) have been introduced in the literature. In these formulations, each intermediate result is represented by a linear function with a small remainder interval (Nedialkov et al., 2004). According to the philosophy of the *affine arithmetic*, Muscolino and Sofi (2011) proposed the so-called *improved interval analysis* based on the definition of the *extra symmetric unitary interval* (EUI) variable $\hat{e}^I\triangleq[-1,+1]$, $(i=1,2,\dots,r)$. The EUI is defined in such a way that the following properties hold:

$$\hat{e}_i^I-\hat{e}_i^I=0;\qquad \hat{e}_i^I\times\hat{e}_i^I=\left(\hat{e}_i^I\right)^2=[1,1]; \quad (38)$$

$$\hat{e}_i^I\times\hat{e}_j^I=[-1,+1],\quad i\ne j;\qquad \hat{e}_i^I\,/\,\hat{e}_i^I=[1,1]. \quad (39)$$

where the subscript *i* means that the EUI variable is associated to the *i*-th uncertain-but-bounded parameter. In the previous equations, [1,1] = 1 is the so-called unitary *thin interval*. It is useful to remember that a thin interval occurs when $\underline{\alpha}=\overline{\alpha}$ and it is defined as $\alpha^I\triangleq[\underline{\alpha},\underline{\alpha}]$, so that $\alpha\in\mathbb{R}$. Then, introducing the midpoint value (or mean), $\alpha_{0,i}$, and the deviation amplitude (or radius), $\Delta\alpha_i$, of the *i*-th real interval variable α_i^I :

$$\alpha_{0,i} = \frac{1}{2}\left(\underline{\alpha}_i + \overline{\alpha}_i\right); \quad \Delta\alpha_i = \frac{1}{2}\left(\overline{\alpha}_i - \underline{\alpha}_i\right), \tag{40}$$

the following *affine form* definition can be adopted:

$$\alpha_i^I = \alpha_{0,i} + \Delta\alpha_i\, \hat{e}_i^I, \quad \left(i = 1, 2, ..., r\right). \tag{41}$$

In the case of complex interval variables, within the framework of the affine arithmetic, Manson (2005) proposed an approach which allows to take into account the dependency between the real and imaginary components of the complex variables. Conversely, the "ordinary" complex interval analysis assumes that the real and imaginary components are independent. According to the philosophy of the affine arithmetic, a complex interval variable $z_i^I = x_i^I + \mathrm{i}\, y_i^I$ is herein defined as:

$$z_i^I = z_{0,i} + \Delta z_i\, \hat{e}_i^I = \left(x_{0,i} + \mathrm{i}\, y_{0,i}\right) + \left(\Delta x_i + \mathrm{i}\,\Delta y_i\right)\hat{e}_i^I \tag{42}$$

where $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit; $x_{0,i}$ and $y_{0,i}$ are the midpoint values (or means) and Δx_i and Δy_i are the deviation amplitude (or radius) of the real and imaginary part of the complex interval variable, respectively, given by:

$$x_{0,i} = \frac{1}{2}\left(\underline{x}_i + \overline{x}_i\right); \quad y_{0,i} = \frac{1}{2}\left(\underline{y}_i + \overline{y}_i\right); \quad \Delta x_i = \frac{1}{2}\left(\overline{x}_i - \underline{x}_i\right); \quad \Delta y_i = \frac{1}{2}\left(\overline{y}_i - \underline{y}_i\right). \tag{43}$$

4.2. INTERVAL STIFFNESS MATRIX

In structural engineering, the uncertain-but-bounded parameters can be reasonably assumed to posses symmetric deviation amplitude $\overline{\alpha}_i = -\underline{\alpha}_i \equiv \alpha_i$, so that the generic interval variable, according to the improved interval analysis, can be written in *affine form* as:

$$\alpha_i^I = \Delta\alpha_i\, \hat{e}_i^I \tag{44}$$

being $\alpha_{0,i} = 0$ and $\Delta\alpha_i > 0$.

Then, following the interval formalism above introduced, the stiffness matrix $\mathbf{K}(\boldsymbol{\alpha})$ can be expressed as a linear function of the interval variables, i.e.:

$$\mathbf{K}(\boldsymbol{\alpha}) = \mathbf{K}_0 + \Delta\mathbf{K}(\boldsymbol{\alpha}) = \mathbf{K}_0 + \sum_{i=1}^{r} \mathbf{K}_i\, \Delta\alpha_i\, \hat{e}_i^I, \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right] \tag{45}$$

where the matrices $\mathbf{K}_0$ and $\mathbf{K}_i$, of order $n \times n$, have been defined in Eq. (3) and $\Delta\alpha_i$ is the dimensionless fluctuation of the i-th uncertain parameter. Furthermore, by virtue of the decomposition (45) of the stiffness matrix, the following relationship holds:

$$\mathbf{\Omega}^2(\boldsymbol{\alpha}) = \mathbf{\Phi}_0^{\mathrm{T}}\mathbf{K}_0\mathbf{\Phi}_0 + \sum_{i=1}^{r}\mathbf{\Phi}_0^{\mathrm{T}}\mathbf{K}_i\mathbf{\Phi}_0\Delta\alpha_i\,\hat{e}_i^I = \mathbf{\Omega}_0^2 + \sum_{i=1}^{r}\mathbf{\Omega}_i^2\Delta\alpha_i\,\hat{e}_i^I, \quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}},\bar{\boldsymbol{\alpha}}\right] \tag{46}$$

where $\mathbf{\Omega}_i^2$ is the matrix defined in Eq.(8).

4.3. APPROXIMATE INTERVAL MODAL FRF MATRIX

In order to simplify interval computations, the attention is herein focused on small deviation amplitudes of the uncertain-but-bounded parameters, i.e. $|\alpha_i| \ll 1$. Under this assumption, based on the *RSE* in Eq.(37) the *interval modal FRF matrix*, in the most general case of discretized structural systems with uncertain-but-bounded stiffness properties, can be expressed in the following approximate explicit form:

$$\mathbf{H}(\boldsymbol{\alpha},\omega) \approx \mathbf{H}_0(\omega) - \sum_{i=1}^{r}\sum_{\ell=1}^{p_i}\frac{\Delta\alpha_i\,\hat{e}_i^I\lambda_i^{(\ell)}}{1+\Delta\alpha_i\,\hat{e}_i^I\,\lambda_i^{(\ell)}\,b_{i\ell}(\omega)}\,\mathbf{B}_{i\ell}(\omega), \quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}},\bar{\boldsymbol{\alpha}}\right] \tag{47}$$

where $b_{i\ell}(\omega)$ and $\mathbf{B}_{i\ell}(\omega)$ are the complex functions defined in Eq.(33). Alternatively, the matrix $\mathbf{H}(\boldsymbol{\alpha},\omega)$ can be rewritten in a more suitable affine form, as follows:

$$\mathbf{H}(\boldsymbol{\alpha},\omega) = \mathbf{H}_0(\omega) + \sum_{i=1}^{r}\sum_{\ell=1}^{p_i}\left(a_{0,i\ell}(\omega) + \Delta a_{i\ell}(\omega)\,\hat{e}_i^I\right)\mathbf{B}_{i\ell}(\omega), \quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}},\bar{\boldsymbol{\alpha}}\right] \tag{48}$$

where $a_{0,i\ell}(\omega)$ and $\Delta a_{i\ell}(\omega)$ are complex functions describing the midpoint and the deviation amplitude of the $i\ell$-th term in Eq.(47), given, respectively, by:

$$a_{0,i\ell}(\omega) = \frac{\left(\Delta\alpha_i\,\lambda_i^{(\ell)}\right)^2 b_{i\ell}(\omega)}{1-\left(\Delta\alpha_i\,\lambda_i^{(\ell)}b_{i\ell}(\omega)\right)^2}; \quad \Delta a_{i\ell}(\omega) = \frac{\Delta\alpha_i\,\lambda_i^{(\ell)}}{1-\left(\Delta\alpha_i\,\lambda_i^{(\ell)}b_{i\ell}(\omega)\right)^2}. \tag{49}$$

Equation (48) can be recast in the following form:

$$\mathbf{H}(\boldsymbol{\alpha},\omega) = \mathbf{N}_0(\omega) + \Delta\mathbf{N}(\boldsymbol{\alpha},\omega), \quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}},\bar{\boldsymbol{\alpha}}\right] \tag{50}$$

where $\mathbf{N}_0(\omega)$ and $\Delta\mathbf{N}(\boldsymbol{\alpha},\omega)$ are the midpoint and the deviation matrices of the modal *FRF* defined in the context of the proposed *RSE*, respectively, as:

$$\mathbf{N}_0(\omega) = \mathbf{H}_0(\omega) + \sum_{i=1}^{r}\sum_{\ell=1}^{p_i}a_{0,i\ell}(\omega)\,\mathbf{B}_{i\ell}(\omega); \tag{51}$$

$$\Delta \mathbf{N}(\boldsymbol{\alpha},\omega) = \sum_{i=1}^{r}\sum_{\ell=1}^{p_i} \Delta a_{i\ell}(\omega) \hat{e}_i^I \, \mathbf{B}_{i\ell}(\omega), \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right]. \tag{52}$$

4.4. BOUNDS OF THE MODULUS OF THE NODAL INTERVAL FRF

The aim of this section is to determine the range of the modulus of the nodal interval *FRFs* of linear discretized structures with uncertain-but-bounded parameters. Once the modal *FRF* matrix is known, the square modulus of the *FRF* of the *p*-th DOF of the structural system can be defined as:

$$\left\| H_{N,pp}(\boldsymbol{\alpha},\omega) \right\|^2 = \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \, \mathbf{H}^*(\boldsymbol{\alpha},\omega) \, \boldsymbol{\phi}_{0,p} \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \, \mathbf{H}(\boldsymbol{\alpha},\omega) \, \boldsymbol{\phi}_{0,p}, \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right] \tag{53}$$

where $\boldsymbol{\phi}_{0,p}^{\mathrm{T}}$ is the *p*-th row of the modal matrix $\boldsymbol{\Phi}_0$ solution of the eigenproblem (5). Substituting Eq. (50) into Eq. (53), the following relationship is obtained:

$$\left\| H_{N,pp}(\boldsymbol{\alpha},\omega) \right\|^2 = \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \left[\mathbf{N}_0^*(\omega) + \Delta \mathbf{N}^*(\boldsymbol{\alpha},\omega) \right] \boldsymbol{\phi}_{0,p} \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \left[\mathbf{N}_0(\omega) + \Delta \mathbf{N}(\boldsymbol{\alpha},\omega) \right] \boldsymbol{\phi}_{0,p}, \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right]. \tag{54}$$

Aiming to evaluate the upper bound and the lower bound of the modulus of $H_{N,pp}(\boldsymbol{\alpha},\omega)$, Eq. (54) is rewritten as:

$$\left\| H_{N,pp}(\boldsymbol{\alpha},\omega) \right\|^2 = \mathrm{mid} \left\| H_{N,pp}(\omega) \right\|^2 + \mathrm{dev} \left\| H_{N,pp}(\boldsymbol{\alpha},\omega) \right\|^2, \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right] \tag{55}$$

where the symbols $\mathrm{mid}\|\bullet\|^2$ and $\mathrm{dev}\|\bullet\|^2$ denote the midpoint and the deviation of the square modulus of the interval nodal *FRF* defined in Eq.(54).

In order to simplify interval computations, higher-order terms are neglected, namely the term $\boldsymbol{\phi}_{0,p}^{\mathrm{T}} \Delta \mathbf{N}^*(\boldsymbol{\alpha},\omega) \boldsymbol{\phi}_{0,p} \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \Delta \mathbf{N}(\boldsymbol{\alpha},\omega) \boldsymbol{\phi}_{0,p}$ in Eq. (54) is disregarded. According to this approximation, the midpoint and the deviation functions introduced in Eq.(55) can be written as:

$$\mathrm{mid} \left\| H_{N,pp}(\omega) \right\|^2 \approx \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \, \mathbf{N}_0^*(\omega) \, \boldsymbol{\phi}_{0,p} \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \, \mathbf{N}_0(\omega) \, \boldsymbol{\phi}_{0,p}; \tag{56}$$

$$\mathrm{dev} \left\| H_{N,pp}(\boldsymbol{\alpha},\omega) \right\|^2 \approx \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \, \mathbf{N}_0^*(\omega) \boldsymbol{\phi}_{0,p} \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \, \Delta \mathbf{N}(\boldsymbol{\alpha},\omega) \, \boldsymbol{\phi}_{0,p} + \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \, \Delta \mathbf{N}^*(\boldsymbol{\alpha},\omega) \, \boldsymbol{\phi}_{0,p} \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \, \mathbf{N}_0(\omega) \, \boldsymbol{\phi}_{0,p}, \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right] \tag{57}$$

where $\mathbf{N}_0(\omega)$ and $\Delta \mathbf{N}(\boldsymbol{\alpha},\omega)$ are the midpoint and the deviation matrices introduced in Eqs.(51) and (52).

The lower bound, $\left\| \underline{H}_{N,pp}(\omega) \right\|^2$, and the upper bound, $\left\| \overline{H}_{N,pp}(\omega) \right\|^2$, of the square modulus of the nodal *FRF* of the *p*-th DOF can be evaluated, according to the philosophy of the affine arithmetic, as the minimum and maximum of the various combinations, i.e.:

$$\left\| \underline{H}_{N,pp}(\omega) \right\|^2 = \text{mid}\left\| H_{N,pp}(\omega) \right\|^2 - \Delta\left\| H_{N,pp}(\omega) \right\|^2 ; \tag{58}$$

$$\left\| \bar{H}_{N,pp}(\omega) \right\|^2 = \text{mid}\left\| H_{N,pp}(\omega) \right\|^2 + \Delta\left\| H_{N,pp}(\omega) \right\|^2 . \tag{59}$$

In the previous equations the function $\Delta\left\| H_{N,pp}(\omega) \right\|^2$ is obtained upon substituting the matrix $\Delta\mathbf{N}(\boldsymbol{\alpha},\omega)$, defined in Eq.(52), into Eq.(57) and then deriving the maximum of the deviation $\text{dev}\left\| H_{N,pp}(\boldsymbol{\alpha},\omega) \right\|^2$ according to the main properties of the interval analysis, i.e.:

$$\Delta\left\| H_{N,pp}(\omega) \right\|^2 = \sum_{i=1}^{r} \left| \sum_{\ell=1}^{p_i} \boldsymbol{\phi}_{0,p}^{\mathrm{T}} \left[\mathbf{N}_0^*(\omega)\boldsymbol{\phi}_{0,p}\boldsymbol{\phi}_{0,p}^{\mathrm{T}} \Delta a_{i\ell}(\omega)\mathbf{B}_{i\ell}(\omega) + \Delta a_{i\ell}^*(\omega)\mathbf{B}_{i\ell}^*(\omega)\boldsymbol{\phi}_{0,p}\boldsymbol{\phi}_{0,p}^{\mathrm{T}}\,\mathbf{N}_0(\omega) \right] \boldsymbol{\phi}_{0,p} \right|. \tag{60}$$

Notice that the function in square brackets is a real function and that the symbol $|\bullet|$ means absolute value. Obviously, the lower bound and the upper bound, $\left\| \underline{H}_{N,pp}(\omega) \right\|$ and $\left\| \bar{H}_{N,pp}(\omega) \right\|$, of the modulus of the nodal *FRF* of the *p*-th DOF can be obtained straightforwardly by taking the square root of Eqs. (58) and (59).

5. Numerical applications

5.1. TRUSS STRUCTURE WITH UNCERTAIN YOUNG'S MODULI

The first numerical application concerns the 24-bar truss structure depicted in Fig. 1. The Young's moduli of $r = 7$ bars are taken as uncertain parameters with fluctuations $|\alpha_i| < 1$ around the nominal value $E_0 = 2.1 \times 10^8$ kN/m^2, i.e. $E_i = E_0\,(1+\alpha_i)$, (i = 18, 19,..., 24). The cross-sectional areas of the bars are set equal to $A_i = 5 \times 10^{-4}$ m^2 while the lengths L_i (i = 1, 2,..., 24) can be deduced from Fig.1 where $L = 3$ m. Furthermore, each node possesses a lumped mass $M = 500$ Kg. Only the first $m = 8$ vibrations modes are retained in the modal analysis and the modal damping ratio has been assumed equal to $\zeta = 0.05$ for all the modes.

In Fig. 2, the exact *FRF* of the first modal coordinate, $H_{11}(\boldsymbol{\alpha},\omega)$, evaluated performing the inversion of the matrix into square brackets in Eq.(11) for $\alpha_i = \alpha = 0.05$, (i = 18, 19,..., 24) is compared with the corresponding approximate *FRF* obtained by applying the proposed *RSE* (Eq.(27)). Notice that a good matching of the exact *FRF* is achieved by retaining only the first-order terms in the *RSE*.

Figure 3 displays an analogous comparison for larger parameter fluctuations, say $\alpha_i = \alpha = 0.1$. As expected, in this case the proposed *RSE* truncated to first-order terms is less accurate, especially in the frequency range around the fundamental frequency of the system. Including second-order terms allows to improve the accuracy, as shown in the enlargement in Fig 3b.

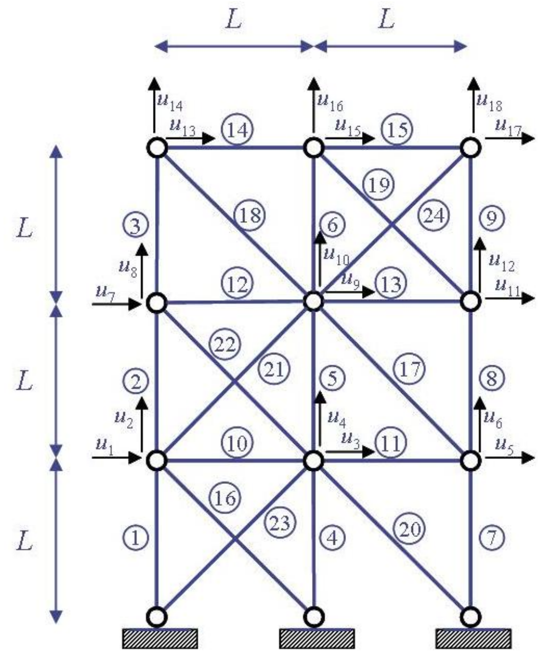

Figure 1. Truss structure with uncertain Young's moduli.

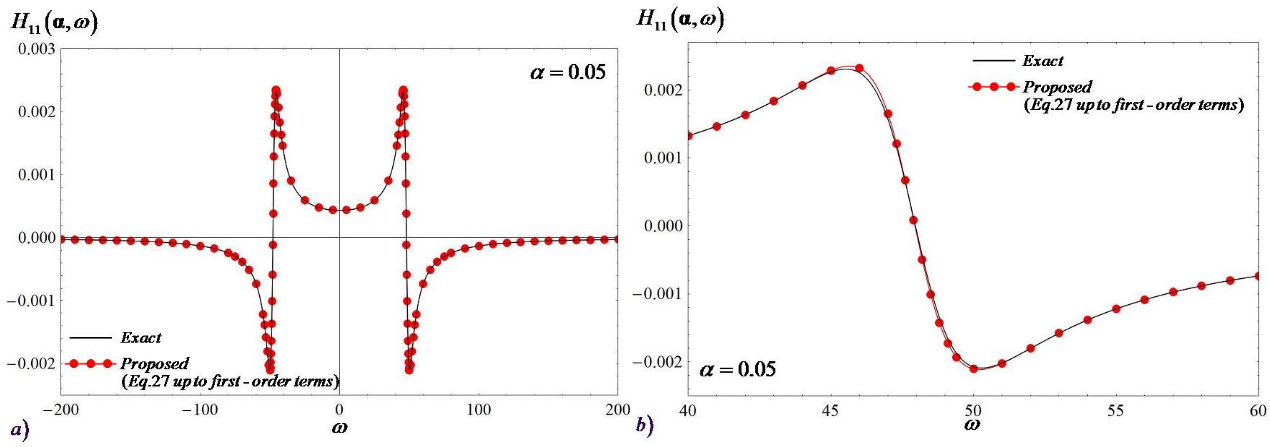

Figure 2. *FRF* of the first modal coordinate $H_{11}(\boldsymbol{\alpha},\omega)$: a) comparison between the exact *FRF* and the proposed *RSE* truncated to first-order terms; b) enlargement showing the convergence of the *RSE* close to the fundamental frequency of the structure ($\alpha = 0.05$).

In order to demonstrate the capability of the proposed explicit approximation of the *FRF* matrix to handle different uncertainty models, the fluctuating Young's moduli of the bars are now treated as interval variables i.e. $E_i^I = E_0(1+\Delta\alpha_i\,\hat{e}_i^I)$, ($i$ = 18, 19,..., 24), with symmetric deviations $\Delta\alpha_i = \Delta\alpha = 0.05$.

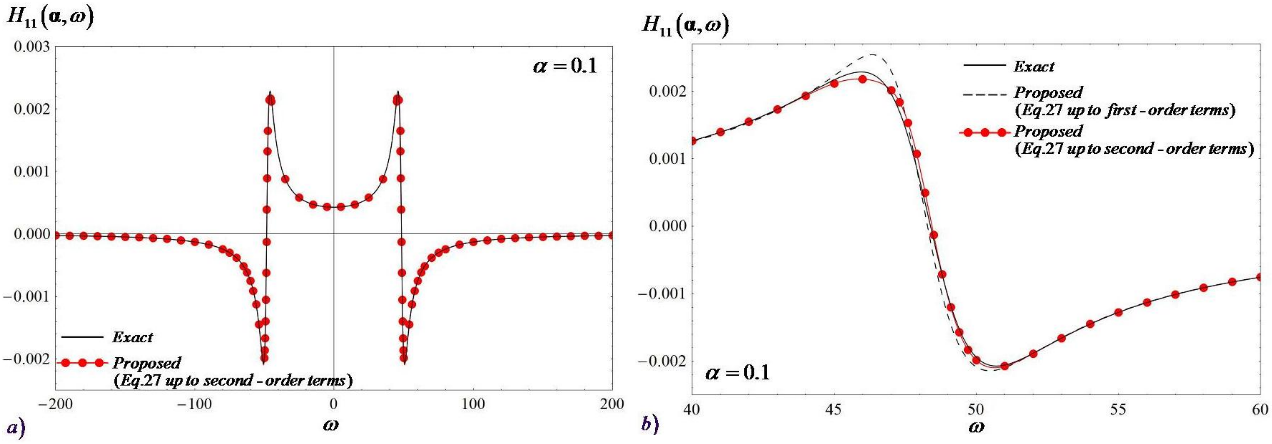

Figure 3. *FRF* of the first modal coordinate $H_{11}(\boldsymbol{\alpha},\omega)$: a) comparison between the exact *FRF* and the proposed *RSE* truncated to second-order terms; b) enlargement showing the convergence of the *RSE* close to the fundamental frequency of the structure ($\alpha = 0.1$).

In Figs.4 and 5, the upper bound and the lower bound of the modulus of the *FRF* of the nodal displacements u_1 and u_{13} of the truss (see Fig. 1), obtained by applying the proposed *RSE* truncated to first-order terms (see Eqs. (58) and (59)), are contrasted with the exact bounds. The latter are obtained following the philosophy of the *vertex method* (Muhanna and Mullen, 2001; Moens and Vandepitte, 2005), namely evaluating the modulus of the *FRF* for all the combinations of the bounds of the uncertain parameters and then taking at each frequency ω the maximum and minimum value among all the moduli of the *FRF* so obtained. Notice that the proposed estimates of the upper bound and lower bound of both $\|H_{N,11}(\boldsymbol{\alpha},\omega)\|$ and $\|H_{N,1313}(\boldsymbol{\alpha},\omega)\|$ are very close to the exact ones.

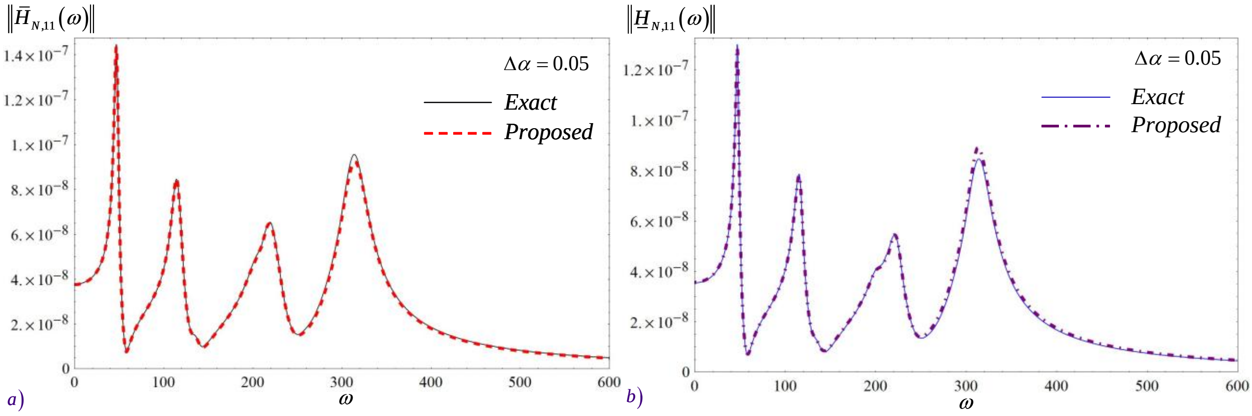

Figure 4. Comparison between the exact and proposed a) upper bound and b) lower bound of the modulus of the *FRF* of the nodal displacement u_1 ($\Delta\alpha = 0.05$).

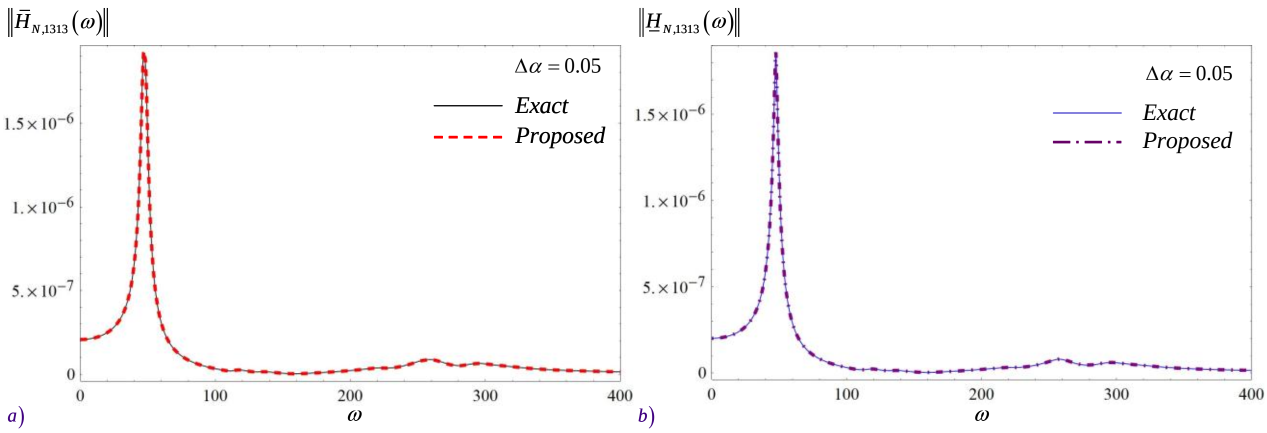

Figure 5. Comparison between the exact and proposed a) upper bound and b) lower bound of the modulus of the *FRF* of the nodal displacement u_{13} ($\Delta\alpha = 0.05$).

5.2. FLEXIBLE FRAME WITH UNCERTAIN YOUNG'S MODULI

As second application, a portal frame with uncertain Young's moduli is considered (see Fig. 6). It is assumed that the elastic moduli of the beam and columns exhibit fluctuations $|\alpha_i| < 1$ around the nominal value $E_0 = 2.85 \times 10^7$ kN/m^2, i.e. $E_i = E_0\,(1+\alpha_i)$, (i = 1, 2, 3). The geometrical properties of the portal frame are indicated in Fig.6, where $b = 0.30$ m, $h = 0.60$ m, $L = 3$ m and $H = 2$ m. Furthermore, each node possesses a lumped mass M=500 Kg. The modal damping ratio is set equal to $\zeta = 0.05$.

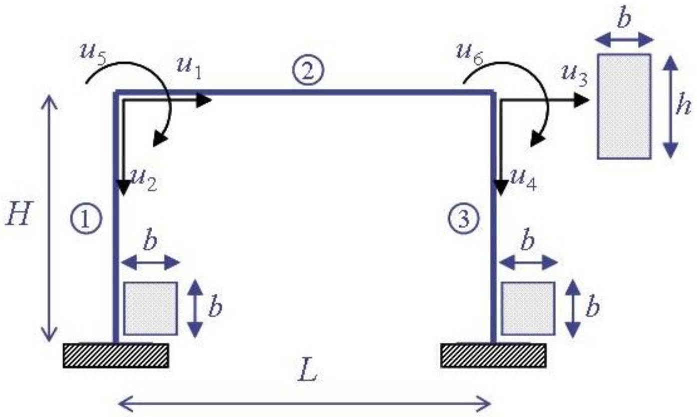

Figure 6. Portal frame with uncertain Young's moduli.

Figure 7 displays the comparison between the exact and approximate *FRFs* of the first modal coordinate, $H_{11}(\boldsymbol{\alpha},\omega)$, for $\alpha_i = \alpha = 0.05$, (i = 1, 2, 3). The convergence of the *RSE* can be detected by inspection of the enlargement in Fig. 6b, where different approximations obtained retaining terms up to the third-order are reported. It can be seen that the proposed *RSE* truncated to the third-order provides an accurate approximation of the *FRF* close to the fundamental frequency of the structure. The results pertaining to larger parameter fluctuations, $\alpha_i = \alpha = 0.10$, shown in Figure 8, demonstrate the accuracy of the proposed *RSE* even for high uncertainty levels. Obviously, in this case higher-order terms of the *RSE* play an increasing important role.

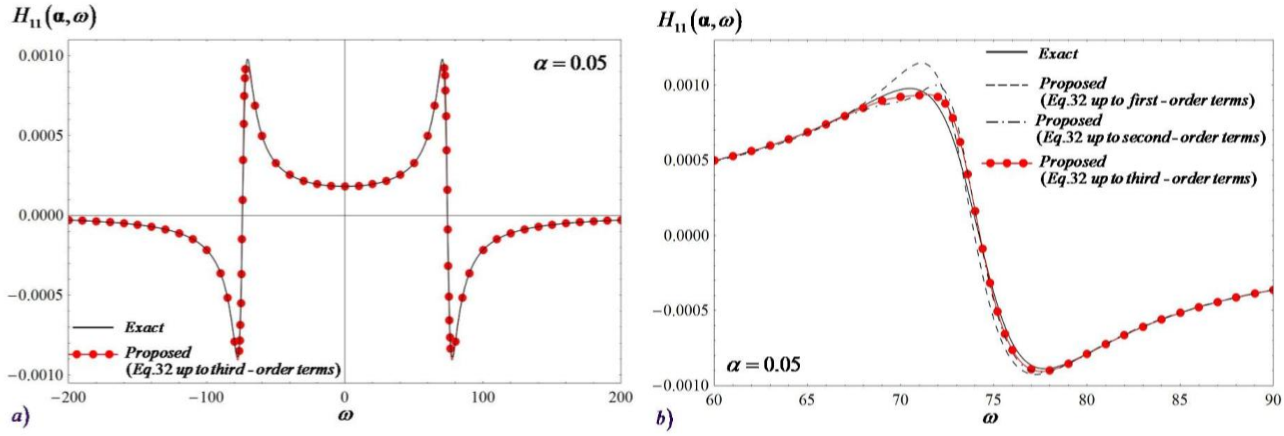

Figure 7. FRF of the first modal coordinate $H_{11}(\boldsymbol{\alpha},\omega)$: a) comparison between the exact *FRF* and the proposed *RSE* truncated to third-order terms; b) enlargement showing the convergence of the *RSE* close to the fundamental frequency of the structure ($\alpha = 0.05$).

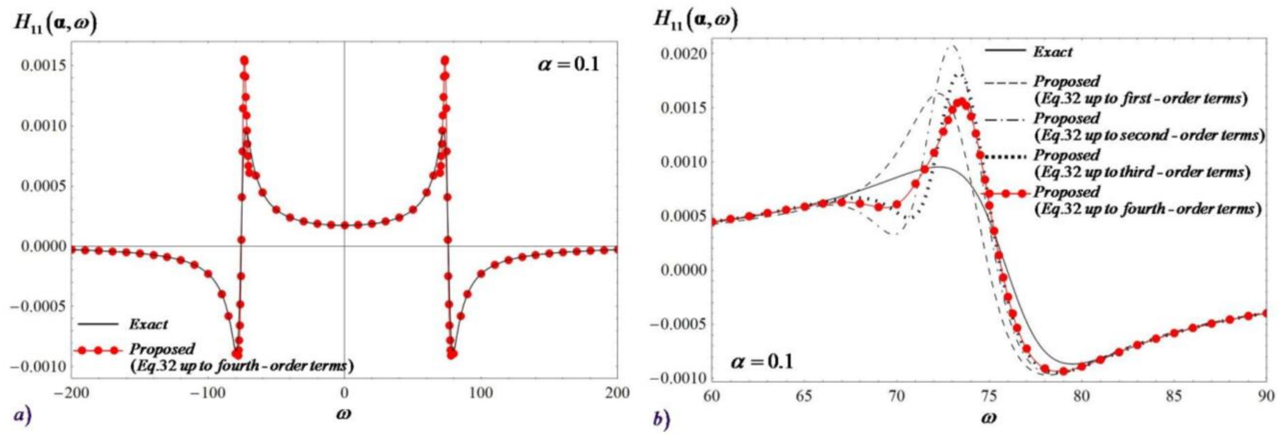

Figure 8. FRF of the first modal coordinate $H_{11}(\boldsymbol{\alpha},\omega)$: a) comparison between the exact *FRF* and the proposed *RSE* truncated to fourth-order terms; b) enlargement showing the convergence of the *RSE* close to the fundamental frequency of the structure ($\alpha = 0.10$).

Finally, the fluctuating Young's moduli of the beam and columns are modelled as uncertain-but-bounded parameters i.e. $E_i^I = E_0(1+\Delta\alpha_i\,\hat{e}_i^I)$, ($i = 1, 2, 3$), with symmetric deviations $\Delta\alpha_i = \Delta\alpha = 0.05$. Figure 9 displays the comparison between the upper bound and the lower bound of the modulus of the *FRF* of the nodal displacement u_1, $\|H_{N,11}(\boldsymbol{\alpha},\omega)\|$, obtained by applying the proposed *RSE* truncated to first-order terms (see Eqs. (58) and (59)), and the exact bounds evaluated following the philosophy of the *vertex method*. It can be observed that also in the case of flexible frames the *RSE* provides accurate estimates of the upper bound and the lower bound of the modulus of the *FRF*.

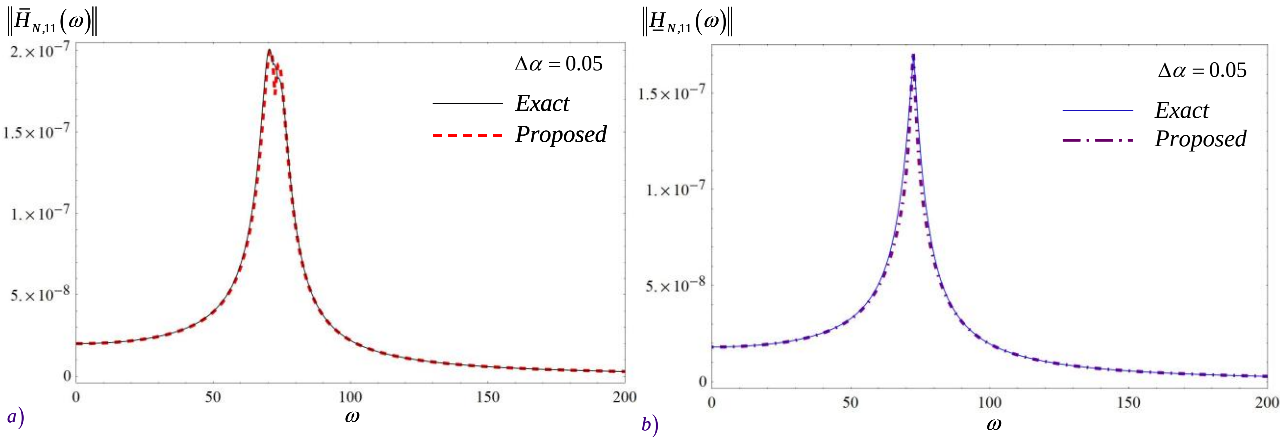

Figure 9. Comparison between the exact and proposed (a) upper bound and (b) lower bound of the modulus of the *FRF* of the nodal displacement u_1 ($\Delta\alpha = 0.05$).

6. Concluding remarks

The evaluation of the *frequency response function* (*FRF*) matrix of linear structures with uncertain stiffness properties has been addressed. Specifically, a procedure for deriving the *FRF* matrix in explicit approximate form has been presented. The proposed method relies on the spectral decomposition of the deviation of the stiffness matrix (with respect to its nominal value) which allows to obtain a sum of rank-one matrices, each one associated to a single uncertain parameter. Then, the equations of motion are projected in the modal subspace and, after some algebra, the Neumann series expansion of the *FRF* matrix is rewritten in an alternative explicit form, herein called *Rational Series Expansion* (*RSE*). The proposed *RSE* represents a useful tool for performing the frequency domain analysis of linear structures with uncertain parameters since it enables one to derive closed form expressions of the response and then investigate the effects of the fluctuating parameters. The latter can be modeled resorting either to probabilistic or non-probabilistic approaches depending on the available information on their variability.

The accuracy of the proposed *RSE* has been assessed by analyzing a truss structure and a portal frame with uncertain Young's moduli. Numerical results have shown that the estimates of the *FRF* provided by the *RSE* are very close to the exact ones even for large fluctuations of the uncertain parameters. The versatility of the proposed *RSE* has been demonstrated by modeling the fluctuating Young's moduli as uncertain-but-bounded parameters. The estimates of the upper bound and lower bound of the modulus of the *FRF* derived by applying the *RSE* in conjunction with the so-called *improved interval analysis* have been shown to be in good agreement with the exact bounds evaluated following the philosophy of the *vertex method*.

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