Nonlinear Interval Finite Elements for Structural Mechanics Problems

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Abstract: Interval Finite Element Method (IFEM) has been developed to handle load, material, and geometric uncertainties that are introduced in a form of interval numbers defined by their lower and upper bounds. However, the scope of the previous methods was limited to linear problems. The present work introduces an IFEM formulation for problems involving material nonlinearity. The algorithm is based on the previously developed high accuracy interval solutions. Two approaches are introduced; an iterative method that generates successive approximations to the secant stiffness and a modified Newton-Raphson method based on deterministic/interval two face strategy that carries out the iteration successfully by identifying interval multipliers for each load throughout the iteration procedure. Examples are presented to illustrate the behavior of both formulations.

Keywords: Finite Elements; Interval; Nonlinear; Materials.

1. Introduction

Structural analysis without considering uncertainty in loading or material properties leads to an incomplete understanding of the structural performance. Structural analysis using interval variables has been used by several researchers to incorporate uncertainty into structural analysis (Koyluoglu, H. U., Cakmak, A. S., and Nielson, S. R. K. 1995, Muhanna, R. L. and Mullen, R. L. 1995, Nakagiri S. and Yoshikawa, N. 1996, Rao, S. S. and Sawyer, P. 1995, Rao, S. S. and Berke, L. 1997, Rao, S.S., and Chen Li 1998, Muhanna and Mullen, 2001, Neumaier and Pownuk 2007). To the authors' knowledge, applications of interval methods for the analysis of structures with material nonlinearity do not exist anywhere in literature.

In this paper, we present an initial investigation into the application of interval finite element methods to non-linear problems of structural mechanics. In this work, we will consider deformation theory of two dimensional truss structures with a plasticity model for the material response. Critical to our development is the computation of element strains with minimal possible overestimation. Usually, derived quantities in Interval Finite Element Method (IFEM) such as stresses and strains have additional overestimation in comparison with primary quantities such as displacements. This issue has plagued displacement-based IFEM for quite some time. The recent development of mixed/hybrid IFEM formulation by the authors (Rama Rao, Mullen and Muhanna, 2010) is capable of simultaneous calculation of interval strains and

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displacements with the same accuracy. This opened the road for further progress in new application areas such as nonlinear analysis.

This work presents two approaches to the solution of interval finite elements with material nonlinearity, namely: the interval secant and interval modified Newton-Raphson methods. The paper is structured as follows. First, a short review of some interval concepts and an overview of linear IFEM are introduced. The interval secant and interval Newton-Raphson methods are then presented in sections 3 and 4. Examples are finally presented and discussed.

2. Linear Interval Finite Element Method

Finite element method is one of the most common numerical methods for solving differential and partial differential equations with enormous applications in different fields of science and engineering. Interval finite element methods have been developed to handle the analysis of systems for which uncertain parameters are described as intervals. A variety of solution techniques have been proposed for IFEM. A comprehensive review of these techniques can be found in (Muhanna *et al.*, 2007, Zhang, 2005, and Rama Rao, Mullen and Muhanna, 2010). Interval analysis concerns the numerical computations involving interval numbers. *All interval quantities will be introduced in non-italic boldface font*. The four elementary operations of real arithmetic, namely addition (+), subtraction (−), multiplication (×) and division (÷) can be extended to intervals. Operations $\mathrm{o} \in \{+,-,\times,\div\}$ over interval numbers $\mathbf{x}$ and $\mathbf{y}$ are defined by the general rule (Moore, 1966; Neumaier, 1990)

$$\mathbf{x} \circ \mathbf{y} = [\min(x \circ y), \max(x \circ y)] \quad for \ \circ \in \{+,-,\times,\div\}, \tag{1}$$

in which x and y denote generic elements $x \in \mathbf{x}$ and $y \in \mathbf{y}$. Software and hardware support for interval computation are available such as (Sun microsystems, 2002; Knüppel, 1994, and INTLAB,1999). For a real-valued function $f(x_1,\ldots,x_n)$, the *interval extension* of $f(\ )$ is obtained by replacing each real variable x_i by an interval variable $\mathbf{x}_i$ and each real operation by its corresponding interval arithmetic operation. From the fundamental property of *inclusion isotonicity* (Moore, 1966), the range of the function $f(x_1,\ldots, x_n)$ can be rigorously bounded by its interval extension function

$$f(\mathbf{x}_1,..,\mathbf{x}_n) \supseteq \{f(x_1,..,x_n) \mid x_1 \in \mathbf{x}_1,..,x_n \in \mathbf{x}_n\} \tag{2}$$

Equation (2) indicates that $f(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ contains the range of $f(x_1,\ldots,x_n)$ for all $x_i \in \mathbf{x}_i$. A natural idea to implement interval FEM is to apply the interval extension to the deterministic FE formulation. Unfortunately, such a naïve use of interval analysis in FEM yields meaningless and overly wide results (Muhanna and Mullen, 2001; Dessombz *et al.*, 2001). The reason is that in interval arithmetic each occurrence of an interval variable is treated as a different, independent variable. It is critical to the formulation of the interval FEM that one identifies the dependence between the interval variables and prevents the overestimation of the interval width of the results. In this paper, an element-by-element (EBE) technique is utilized for element assembly (Muhanna and Mullen, 2001; Zhang, 2005). The elements are detached so that there are no connections between elements, avoiding element coupling. The Lagrange multiplier method is then employed to impose constraints to ensure the compatibility. Then a mixed/hybrid formulation is incorporated to simultaneously calculate the interval strains and displacements (Rama Rao, Mullen and Muhanna, 2010). This linear formulation results in the interval linear system of equations that has the following structure:

$$(K + B\,\mathbf{D}\,A)\,\mathbf{u} = a + F\,\mathbf{b}, \tag{3}$$

with interval quantities in **D** and **b** only. The term ($K + B\,\mathbf{D}\,A$) represents the interval structural stiffness matrix and the $a + F\,\mathbf{b}$ term, the structural loading. Any interval solver can be used to solve Eq. (3), however, the following iterative scheme that is developed by Neumaier (Neumaier and Pownuk, 2007) is superior for large uncertainty, defining:

$$C := (K + BD_0A)^{-1} \tag{4}$$

where D_0 is chosen in a manner that ensures its invertability (often D_0 is selected as the midpoint of **D**), the solution **u** can be written as:

$$\mathbf{u} = (Ca) + (CF)\mathbf{b} + (CB)\mathbf{d} \tag{5}$$

To obtain a solution with tight interval enclosure we define two auxiliary interval quantities,

$$\begin{aligned} \mathbf{v} &= A\mathbf{u} \\ \mathbf{d} &= (D_0 - \mathbf{D})\mathbf{v}, \end{aligned} \tag{6}$$

which, given an initial estimate for **u**, we iterate as follows:

$$\mathbf{v}^{k+1} = \{ACa) + (ACF)\mathbf{b} + (ACB)\mathbf{d}^k\} \cap \mathbf{v}^k, \quad \mathbf{d}^{k+1} = \{(D_{c0} - \mathbf{D}_c)\mathbf{v}^{k^{+1}} \cap \mathbf{d}^k, \tag{7}$$

until the enclosures converge, from which the desired solution **u** can be obtained in a straightforward manner.

In this paper the above mentioned iterative enclosure has been used for the solution of the linear interval system of Equation (3). The solution includes displacements, strains, and forces simultaneously with the same high level of accuracy.

3. Interval Secant Method

The first method chosen for solving the system of non-linear interval equations resulting from the interval finite element method is the secant method (Cook, 2002). Given a constitutive relationship, the secant method is an iterative approach that predicts the value of the secant modulus corresponding to a certain level of loading. If load uncertainty is given as an interval value, the resulting element strain will also be an interval quantity. This will lead to an interval value for the secant modulus with the bounds on secant modulus calculated from the bounds on the element strain. In the present work, we will introduce an iterative algorithm that allows the prediction of the interval secant modulus and calculates relevant quantities such as stresses, strains, and displacements for nonlinear material problems

To illustrate the approach, we will assume that for each element in the structure the constitutive relationship is defined as a cubic function as shown below:

$$\sigma = a\varepsilon + b\varepsilon^3, \tag{8}$$

as shown in Figure 1, where σ, ε, a, and b are stress, strain, and constants respectively. The iteration process starts by taking the initial value of the secant modulus at zero strain. In subsequent iterations, a secant modulus is calculated from the current element strain using Eq. (8) as

$$Es(i) = \frac{\sigma(i)}{\varepsilon(i)}, \tag{9}$$

where $Es(i)$, $\sigma(i)$, and $\varepsilon(i)$ are the secant modulus, stress, and strain at iteration i respectively. The iterations continue until convergence with respect to the secant modulus is achieved. However, if the load is given as an interval value, the resulting secant modulus will also be an interval quantity.

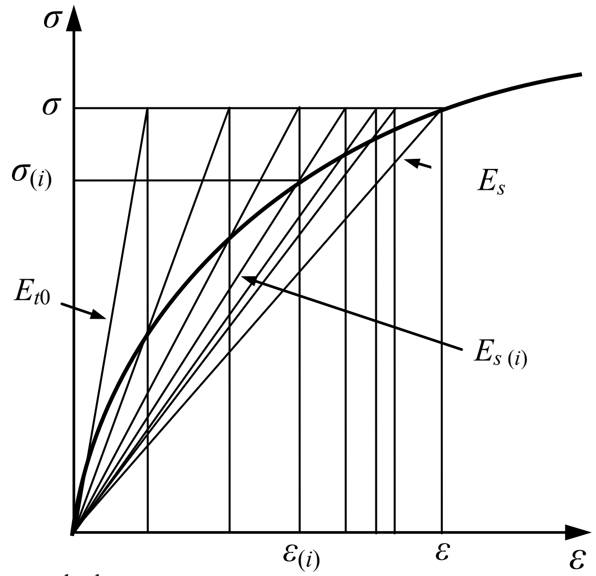

Figure 1. Stress-strain relationship, secant method

A direct calculation of the interval secant modulus from Eq. (9) will lead to an overestimation, however, considering the physics of the problem we can confirm that the lower and upper bounds of the stress in a given element correspond to lower and upper bounds of the strain respectively. Considering this dependency, the interval form of Equation (8) can be introduced as

$$[\underline{Es}(i), \overline{Es}(i)] = [\inf(\frac{\underline{\sigma}(i)}{\underline{\varepsilon}(i)}, \frac{\overline{\sigma}(i)}{\overline{\varepsilon}(i)}), \sup(\frac{\underline{\sigma}(i)}{\underline{\varepsilon}(i)}, \frac{\overline{\sigma}(i)}{\overline{\varepsilon}(i)})] \tag{10}$$

The IFEM formulation presented in section 2 (Rama Rao, Mullen and Muhanna, 2010) provides the strains along with displacements and forces. Solution of conventional interval finite elements provides interval bounds of displacements, the calculations of strains from displacements result in significant overestimation. In the current formulation, the strains are not calculated from displacements but are obtained simultaneously with the displacements and forces, thus avoiding any additional *overestimation*. These sharp bounds on interval strains thus obtained are used in the following iterative algorithm for calculating the updated interval values of the secants.

3.1. ALGORITHM FOR SECANT UPDATE

The following notations are used:

K	:	interval stiffness matrix
P	:	interval load vector
U	:	solution vector, includes stain and stress vectors
ε:	:	current strain
σ	:	current stress
inf:	:	infimum
sup:	:	supremum
$\mathbf{E}_s$	:	current secant
E_{t0}	:	initial secant modulus

for count = 1: countmax
    $\mathbf{K}_c$(U) **U** = **P**
    U = K^{-1} (U) **P** : *Obtain solution based on algorithm given in section* 2.
        for e = 1: number of elements
            max (σ) = $a \times \sup(\boldsymbol{\varepsilon}) + b \times (\sup(\boldsymbol{\varepsilon}))^3$
            max(E_s) = max (σ) / sup (**ε** (e))
            min (σ) = $a \times \inf(\boldsymbol{\varepsilon}) + b \times (\inf(\boldsymbol{\varepsilon}))^3$
            min(E_s) = min (σ) / inf (**ε** (e))
            $\mathbf{E}_s$ (e) = infsup (min (E_s), max (E_s))
        end : of loop on elements
    $\mathbf{K}_c$:   update K with the new values of $\mathbf{E}_s$
end : of loop on count

For the stopping criterion the sum of the L_1 norms of the following relative change of the secant lower and upper bounds is required to be less than a specified small value

$$\left\| \frac{\underline{E}s(i+1) - \underline{E}s(i)}{\underline{E}s(i)} \right\|_1 , \quad \left\| \frac{\overline{E}s(i+1) - \overline{E}s(i)}{\overline{E}s(i)]} \right\|_1 \tag{11}$$

4. Interval Modified Newton-Raphson Method

Newton-Raphson method and modified Newton-Raphson method are iterative methods to find the relation of load versus displacement based on a given constitutive relationship, or within the context of finite elements, to solve the following nonlinear system of equations

$$K(U)U = P, \tag{12}$$

where K, U, and P are the stiffness matrix, the displacement vector, and the load vector, respectively.

Modified Newton-Raphson method in finite element applications uses incremental tangent stiffness at each loading level to predict the displacement as summation of a number of incremental solutions using the out of balance load for each increment. For each increment (iteration) the balance of forces at each node is checked until equilibrium is attained that represents the convergence for that specific load level.

If the applied load is given as an interval value, the internal forces at each node will be intervals, checking the equilibrium at each node will represent a significant challenge. As a matter of fact, the dependency of internal forces on the applied load will result in an overestimation that will not allow equilibrium to be checked properly at each node. In the next section we will introduce a formulation in which we try to delay the use of interval multipliers as much as possible in a way that nodal equilibrium can be checked.

4.1. FORMULATION OF INTERVAL NEWTON-RAPHSON METHOD

The formulation of interval finite element introduced in a recent work by the authors (Rama Rao, Mullen and Muhanna, 2010), provides a solution vector that includes displacements, internal forces, and strains of the system. Using the tangent stiffness, $K_{t,}$ in each of the iterations will convert the system of equations in Equation (12) to a linear system of equations of the form

$$K_t\mathbf{U} = \mathbf{P}, \tag{13}$$

this equation can be reintroduced in the form

$$K_t\mathbf{U} = M\mathbf{d}, \tag{14}$$

where M is a matrix with dimensions (No. degrees of freedom × number of loads) and **d** is a vector of load interval multipliers, (Mullen and Muhanna 1999). The solution of Equation (14) can be given in the following form

$$\mathbf{U} = K_t^{-1}M\mathbf{d}, \tag{15}$$

or

$$\mathbf{U} = M_1\mathbf{d}, \tag{15a}$$

where M_1 is a deterministic matrix with the dimensions (No. degrees of freedom × number of loads). The entries of this matrix are the system solution introduced per each load, or the solution Load-By-Load (LBL). For the clarity of formulation we will assume only two applied loads. This assumption will not impose any restriction on the generality of the formulation. Since we order the unknowns in U as first displacements then element forces, ending with element strains, the entries of last rows of matrix M_1 are element strains introduced LBL and have the form

$$M_s = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \\ \vdots & \vdots \\ \varepsilon_{n1} & \varepsilon_{n2} \end{pmatrix}, \tag{16}$$

where ε_{ij} is the strain of the i^{th} element due to the j^{th} load. Truss structures will be considered for the rest of the paper, such consideration should not limit the generality of the formulation.

In Newton-Raphson iteration and for a given load level, using the tangent stiffness will result in a vector of internal forces different from the vector of applied forces and equilibrium will not be satisfied. The difference between the two vectors is used to compute the residual response until convergence, or equilibrium is attained. This will require the computation of the internal force vector and the comparison with the applied load vector for each of the iterations. In the current interval formulation, as mentioned above, we will try to delay operations on intervals as much as possible.

If we consider the nonlinear constitutive relationship in Equation (8) the internal force for any element can be given as

$$F_i = \sigma_i A_i = (a\varepsilon_i + b\varepsilon_i^3)A_i, \tag{17}$$

where A_i is the cross-sectional area of the i^{th} element. Substituting for strains from equation (16) and including load interval multipliers in Equation (15), we get

$$F_i = \{a\left(\varepsilon_{i1} \quad \varepsilon_{i2}\right)\begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} + b\,[\left(\varepsilon_{i1} \quad \varepsilon_{i2}\right)\begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix}]^3\}A_i, \tag{17}$$

or

$$F_i = [a(\varepsilon_{i1}\mathbf{d}_1 + \varepsilon_{i2}\mathbf{d}_2) + b(\varepsilon_{i1}^3\mathbf{d}_1^3 + \varepsilon_{i2}^3\mathbf{d}_2^3 + 3\varepsilon_{i1}\varepsilon_{i2}^2\mathbf{d}_1\mathbf{d}_2^2 + 3\varepsilon_{i1}^2\varepsilon_{i2}\mathbf{d}_1^2\mathbf{d}_2)]A_i, \tag{18}$$

If n elements meet at node m, the x global component of resultant of element forces can be obtained as

$$\begin{aligned} \mathbf{F}_{mx} = {} & [a(\varepsilon_{11}A_1\cos\theta_1 + \cdots + \varepsilon_{n1}A_n\cos\theta_n)\mathbf{d}_1 + \\ & + a(\varepsilon_{12}A_1\cos\theta_1 + \cdots + \varepsilon_{n2}A_n\cos\theta_n)\mathbf{d}_2 + \\ & + b[(\varepsilon_{11}^3A_1\cos\theta_1 + \cdots + \varepsilon_{n1}^3A_n\cos\theta_n)\mathbf{d}_1^3 + \\ & + \quad (\varepsilon_{12}^3A_1\cos\theta_1 + \cdots + \varepsilon_{n2}^3A_n\cos\theta_n)\mathbf{d}_2^3 + \\ & + (3\varepsilon_{11}\varepsilon_{12}^2A_1\cos\theta_1 + \cdots + 3\varepsilon_{n1}\varepsilon_{n2}^2A_n\cos\theta_n)\mathbf{d}_1\mathbf{d}_2^2 + \\ & + (3\varepsilon_{11}^2\varepsilon_{12}A_1\cos\theta_1 + \cdots + 3\varepsilon_{n1}^2\varepsilon_{n2}A_n\cos\theta_n)\mathbf{d}_1^2\mathbf{d}_2\,] \end{aligned} \tag{19}$$

and both components can be introduced in the following form

$$\begin{aligned}\mathbf{F}_{mx} &= c_{mx1}\mathbf{d}_1 + c_{mx2}\mathbf{d}_2 + c_{mx3}\mathbf{d}_1^3 + c_{mx4}\mathbf{d}_2^3 + c_{mx5}\mathbf{d}_1\mathbf{d}_2^2 + c_{mx6}\mathbf{d}_1^2\mathbf{d}_2 \\ \mathbf{F}_{my} &= c_{my1}\mathbf{d}_1 + c_{my2}\mathbf{d}_2 + c_{my3}\mathbf{d}_1^3 + c_{my4}\mathbf{d}_2^3 + c_{my5}\mathbf{d}_1\mathbf{d}_2^2 + c_{my6}\mathbf{d}_1^2\mathbf{d}_2\end{aligned} \tag{20}$$

or

$$\begin{pmatrix}\mathbf{F}_{mx}\\ \mathbf{F}_{my}\end{pmatrix} = \begin{pmatrix}c_{mx1} & c_{mx2}\\ c_{my1} & c_{my2}\end{pmatrix}\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix} + \begin{pmatrix}c_{mx3} & c_{mx4}\\ c_{my3} & c_{my4}\end{pmatrix}\begin{pmatrix}\mathbf{d}_1^3\\ \mathbf{d}_2^3\end{pmatrix} + \begin{pmatrix}c_{mx5} & c_{mx6}\\ c_{my5} & c_{my6}\end{pmatrix}\begin{pmatrix}\mathbf{d}_1\mathbf{d}_2^2\\ \mathbf{d}_1^2\mathbf{d}_2\end{pmatrix} \tag{21}$$

and the vector of internal forces, after including all nodes, can be introduced in the form

$$\mathbf{F} = \begin{pmatrix}\mathbf{F}_{1x}\\ \mathbf{F}_{1y}\\ \vdots\\ \mathbf{F}_{mx}\\ \mathbf{F}_{my}\end{pmatrix} = \begin{pmatrix}c_{1x1} & c_{1x2}\\ c_{1y1} & c_{1y2}\\ \vdots & \vdots\\ c_{mx1} & c_{mx2}\\ c_{my1} & c_{my2}\end{pmatrix}\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix} + \begin{pmatrix}c_{1x3} & c_{1x4}\\ c_{1y1} & c_{1y2}\\ \vdots & \vdots\\ c_{mx3} & c_{mx4}\\ c_{my3} & c_{my4}\end{pmatrix}\begin{pmatrix}\mathbf{d}_1^3\\ \mathbf{d}_2^3\end{pmatrix} + \begin{pmatrix}c_{1x5} & c_{1x6}\\ c_{1y5} & c_{1y6}\\ \vdots & \vdots\\ c_{mx5} & c_{mx6}\\ c_{my5} & c_{my6}\end{pmatrix}\begin{pmatrix}\mathbf{d}_1\mathbf{d}_2^2\\ \mathbf{d}_1^2\mathbf{d}_2\end{pmatrix} \tag{22}$$

where m is the total number of nodes in the system. Interval internal force vector in Equation (22) is introduced as a product of two separate parts; deterministic and interval. The interval part represents the original load multipliers. This form will allow the comparison of the deterministic values of the applied load (matrix M in Equation (13)) with the deterministic values of the internal forces during the iteration procedure. If we reintroduce Equation (22) in the form

$$\mathbf{F} = MM_1\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix} + MM_2\begin{pmatrix}\mathbf{d}_1^3\\ \mathbf{d}_2^3\end{pmatrix} + MM_3\begin{pmatrix}\mathbf{d}_1\mathbf{d}_2^2\\ \mathbf{d}_1^2\mathbf{d}_2\end{pmatrix} \tag{22}$$

and compare with Equation (13) for the case of two loads

$$P = M_1\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix}, \tag{23}$$

The '*out of balance'* force vector can now be introduced as

$$\delta\mathbf{F} = (M + MM_1)\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix} + MM_2\begin{pmatrix}\mathbf{d}_1^3\\ \mathbf{d}_2^3\end{pmatrix} + MM_3\begin{pmatrix}\mathbf{d}_1\mathbf{d}_2^2\\ \mathbf{d}_1^2\mathbf{d}_2\end{pmatrix} \tag{24}$$

To complete the iteration and update procedure we will introduce what is called deterministic/interval strategy. This strategy is based on introducing a deterministic out of balance Load-By-Load matrix in the form

$$\delta M = (M + MM_1) + MM_2 + MM_3 \tag{24}$$

and providing a deterministic solution in the form

$$\delta UM = K^{-1}\delta M \tag{25}$$

where the entries of δUM are the deterministic solution for a given iteration introduced LBL. The solution will be used to update the deterministic LBL element strains as follows

$$Ms = Ms + \delta UMs \tag{26}$$

where δUMs is a matrix of the dimension (number of elements × number of loads). This matrix is the bottom part of δUM that contains incremental values of element strains listed LBL. The updated value of Ms is used to update the values of internal forces.

On the interval side the incremental solution will be obtained from

$$\delta \mathbf{U} = K^{-1}\delta \mathbf{F} \tag{26}$$

and the interval solution update is

$$\mathbf{U} = \mathbf{U} + \delta \mathbf{U} \tag{27}$$

Crucial to the quality of the solution given in Equation (27) is the evaluation of $\delta\mathbf{F}$ used in Equation (26). For example, if we consider the x component of internal forces at node m as in Equation (20), the '*out of balance'* force can be given as

$$\delta \mathbf{F}_{mx} = (m_{mx} + c_{mx1})\mathbf{d}_1 + c_{mx2}\mathbf{d}_2 + c_{mx3}\mathbf{d}_1^3 + c_{mx4}\mathbf{d}_2^3 + c_{mx5}\mathbf{d}_1\mathbf{d}_2^2 + c_{mx6}\mathbf{d}_1^2\mathbf{d}_2 \tag{28}$$

The objective here is to obtain the tightest possible enclosure for $\delta\mathbf{F}_{mx}$. Due to the multiple occurrences of $\mathbf{d}_1$ and $\mathbf{d}_2$ in (28), a direct evaluation of the function will lead to overestimation due to what is called interval dependency. Special treatment is required to obtain a tight enclosure. This is done by using *inclusion isotonicity* property of Interval arithmetic (Moore *et*, *all*, 2009, Neumaier 1990).

In other words, given a function $f = f(x_1, \ldots, x_n)$ of several variables, the precise range of values taken by f as x_i varies through given intervals $\boldsymbol{x}_i$ is introduced in the form

$$f(\mathbf{x}_1, \cdots, \mathbf{x}_n) = \{f(x_1, \cdots, x_n) : x_1 \in \mathbf{x}_1, \cdots, x_n \in \mathbf{x}_n\} \tag{29}$$

Usually, centered forms are used to reduce overestimation due to dependency of the enclosure of f (Moore

1979, Neumaier 1990). In the present work, the boundary value form (Neumaier, 1990) is used to evaluate the enclosure of function given in Equation (24). The following boundary value form has been suggested by Professor Neumaier during private communication for an enclosure of the function in Equation (28)

$$\begin{aligned}\delta\mathbf{F}_{mx}(d_1, d_2) = {} & [(m_{mx} + c_{mx1})d_{01} + c_{mx2}d_{02}] + (c_{mx3}d_{01} + c_{mx6}d_{02})\mathbf{d}_1^2 + \\ & (c_{mx5}d_{01} + c_{mx4}d_{02})\mathbf{d}_2^2 + (\mathbf{d}_1 - d_{01})[(m_{mx} + c_{mx1}) + c_{mx3}\mathbf{d}_1 + c_{mx5}\mathbf{d}_2^2] + \\ & (\mathbf{d}_2 - d_{02})(c_{mx2} + c_{mx6}\mathbf{d}_1^2 + c_{mx4}\mathbf{d}_2^2)\end{aligned} \tag{30}$$

taking the intersection of the two results computed:

a. with lower bounds in place of d_{01} and d_{02}
b. with upper bounds in place of d_{01} and d_{02}

4.2. Stopping Criteria

Two stopping criteria can be used. The first criterion is deterministic and it is straightforward requiring that

$$\frac{\|\delta M\|_2}{\|M\|_2} \le \delta, \tag{31}$$

where δ is a small specified value. The second criterion is the new containment stopping criterion, which is intrinsic to interval arithmetic. In the deterministic version of modified Newton-Raphson method, the usual stopping criterion is to continue the iteration procedure until the resisting internal forces are equal to the applied loads at each node. In the interval version, the applied loads and internal resisting forces are both intervals and the goal of the iteration is that the interval resisting forces to evolve until become equal to the applied ones. Due to dependency and resulting overestimation it is very difficult to capture such moment of equality between the interval applied loads and interval internal forces. As a matter of fact, during the iteration procedure, the difference between the values of interval internal forces and interval applied loads continues to become smaller converging from one side until a certain stage where one bound of the interval internal forces switches side and contains the interval applied load. A verification of the results of the iteration when the containment occurs shows that the correct solution is indeed obtained. To observe that '*the solution is reached when the evolved interval internal resisting forces contain the interval applied loads'* makes a complete engineering sense. Figure 2 illustrates the containment stopping criterions presented in terms of stress-strain instead of load-displacement.

5 Example Problems

Two example problems are chosen to illustrate the applicability of the present interval approach to handle material nonlinearity in case of truss problems. These examples are chosen to demonstrate the ability of the

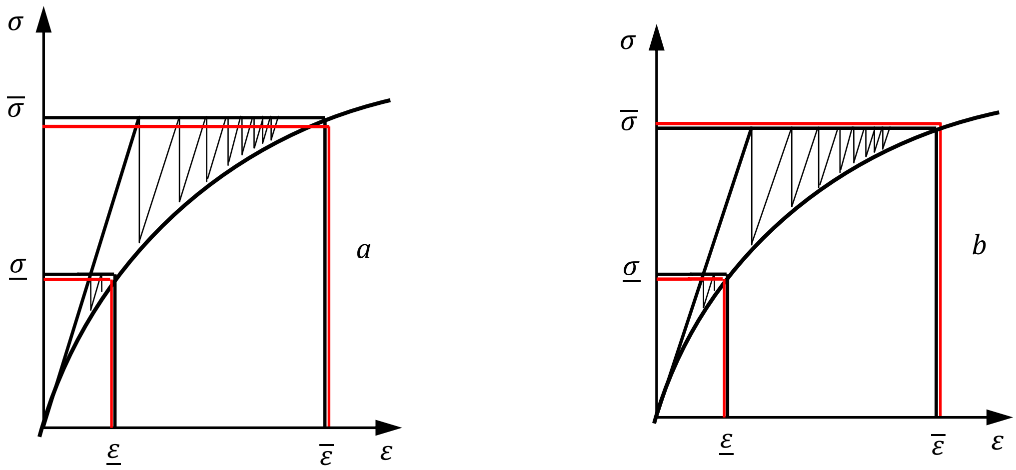

Figure 2. Stress-strain relationship, Modified Newton-Raphson method. *a*) before containment, *b*) after containment

current approach to obtain sharp bounds to the displacements and forces even in the presence of large uncertainties and large number of interval variables. It is assumed that for each element in the structure the constitutive relationship is given in the following cubic functional form for both examples

$$\sigma = E\varepsilon - 10^{13.6}\varepsilon^3, \tag{32}$$

where E is the modulus of elasticity. This relationship is shown in the Figure 3.

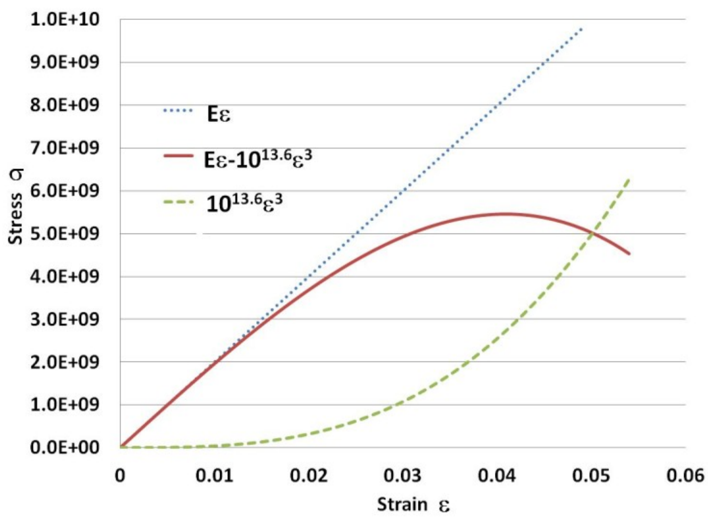

Figure 3. Stress-strain relationship given in Equation (32)

The two example problems are solved for various levels of interval widths of the loads centered at their nominal values. All interval variables are assumed to vary independently. Solution procedures outlined in sections 3 and 4 are based on the current values of member strains ε. These strains can be obtained using three different approaches viz. solution using modified Newton-Raphson method, the secant method and combinatorial approach. The first and second approaches compute member strains with same level of accuracy as displacements. In the third approach, member strains are computed combinatorially in each iteration.

The first example chosen is a five bar truss (Rama Rao, Mullen and Muhanna, 2011) as shown in Figure 4. The truss is subjected to a nominal point load of 200 kN at the node 2 in the horizontal direction to the right and a nominal point load of 200 kN at the node 3 in the vertically downward direction. The Young's modulus of each element is $E_i = 2 \times 10^{11}$ N/m^2, i = 1,5 while the cross sectional area is 1.0×10^{-4} m^2.

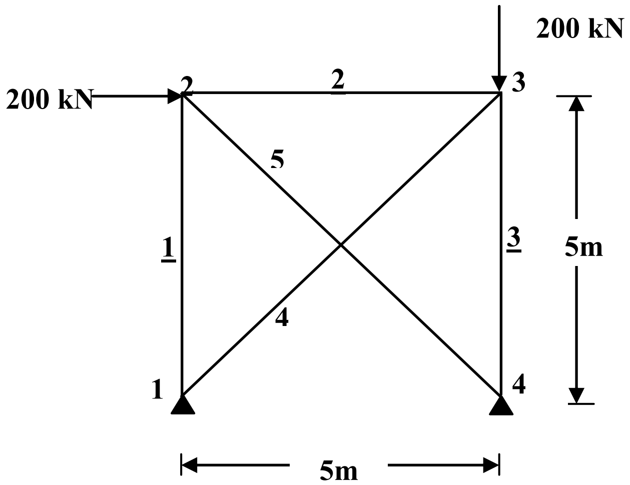

Figure 4. Five bar truss

Tables I, II and III show the computed values of selected displacements (horizontal displacement U_2 at node 2 and vertical displacement V_3 at node 3) and selected strains (strains ε_1 and ε_3 in elements 1 and 3) using the above approaches. Load uncertainties considered in Tables I, II and III are 1%, 10% and 25% (± 0.5%, ± 5% and ± 12.5% respectively about the mean value of load). Overestimation involved in results using the modified Newton-Raphson approach and secant approach is evaluated by comparing the corresponding solutions obtained with the combinatorial approach. Percentage error in the lower and upper bounds of the present solution is computed with reference to the corresponding bounds of the combinatorial solution. It is observed from these tables that error in bounds is quite small for displacements and strains (U_2, V_3, ε_1 and ε_3).

It is observed that the errors in strains (secondary unknowns) are numerically comparable with the error of displacements (primary unknowns). Thus, the present approach succeeds in obtaining the same level of sharpness for primary and derived quantities. This observation holds true at larger values of uncertainty as it will be seen in Tables II and III.

Table I Five bar truss − displacements for 1% uncertainty the load

Method	$U_2 \times 10^1$ (m)		$V_3 \times 10^2$ (m)		$\varepsilon_1 \times 10^3$		$\varepsilon_3 \times 10^2$	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	1.30644	1.32011	-6.92295	-6.84845	6.77652	6.84633	-1.38459	-1.36969
Secant	1.30643	1.32012	-6.92303	-6.84833	6.77554	6.84730	-1.38460	-1.36966
Error%	0.0008	0.0008	0.0012	0.0018	0.0145	0.0142	0.0007	0.0022
Newton	1.30541	1.32113	-6.93144	-6.83991	6.76784	6.85500	-1.38628	-1.36798
Error%	0.0784	0.0774	0.1227	0.1247	0.128	0.127	0.123	0.125

Table II Five bar truss − displacements for 10% uncertainty the load

Method	$U_2 \times 10^1$ (m)		$V_3 \times 10^2$ (m)		$\varepsilon_1 \times 10^3$		$\varepsilon_3 \times 10^2$	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	1.24516	1.38184	-7.26069	-6.51559	6.46303	7.16125	-1.45214	-1.30312
Secant	1.24475	1.38195	-7.26160	-6.51138	6.45189	7.17150	-1.45232	-1.30227
Error%	0.0329	0.0080	0.0125	0.0646	0.1724	0.1431	0.0124	0.0652
Newton	1.23489	1.39212	-7.34664	-6.43065	6.37602	7.24826	-1.46932	-1.28613
Error%	0.8247	0.7444	1.1838	1.3036	1.346	1.215	1.184	1.304

Table III Five bar truss − displacements for 25% uncertainty the load

Method	$U_2 \times 10^1$ (m)		$V_3 \times 10^2$ (m)		$\varepsilon_1 \times 10^3$		$\varepsilon_3 \times 10^2$	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	1.14376	1.48561	-7.83351	-5.96876	5.94287	7.68894	-1.56670	-1.19375
Secant	1.14159	1.48600	-7.83605	-5.94566	5.90929	7.71708	-1.56721	-1.18913
Error%	0.1897	0.0263	0.0324	0.3870	0.5650	0.3660	0.0326	0.3870
Newton	1.11797	1.51263	-8.05715	-5.75784	5.72444	7.91333	-1.61143	-1.15156
Error%	2.2544	1.8189	2.8550	3.5337	3.675	2.918	2.855	3.534

Figure 5 shows the computed interval values of horizontal displacement U_2 at node 2. The figure depicts the variation of the widths of the modified Newton-Raphson, the secant and the combinatorial approaches with the variation of load from its mean value. It is observed from this figure that the solutions computed using tangent and secant approaches enclose the combinatorial solution at all values of variation from 0 percent to 25 percent. A similar behavior is observed in the plot for variation of width of vertical displacement V_3 at node 3 in Figure 6. Figure 7 and 8 show the variation of strains in members 1 and 3 with the variation of uncertainty of load. *It is observed from these figures that the present solution using modified Newton-Raphson approach and secant approach enclose the combinatorial solution at all levels of uncertainty.*

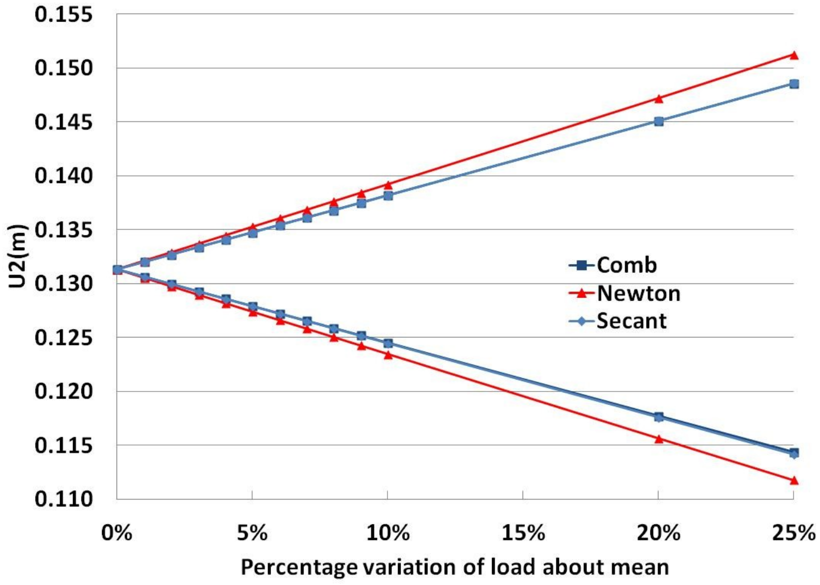

Figure 5. Five bar truss – variation of horizontal displacement at node 2 with uncertainty of load

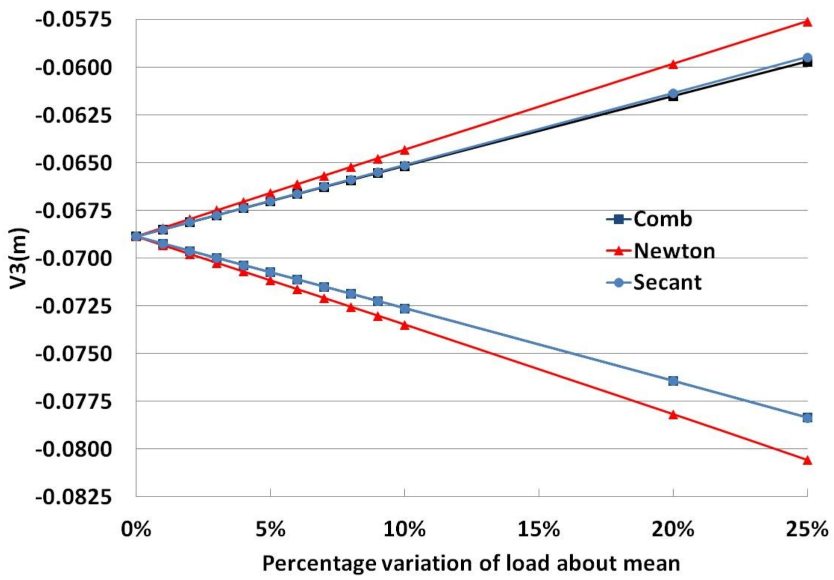

Figure 6. Five bar truss – variation of vertical displacement of node 3 with uncertainty of load

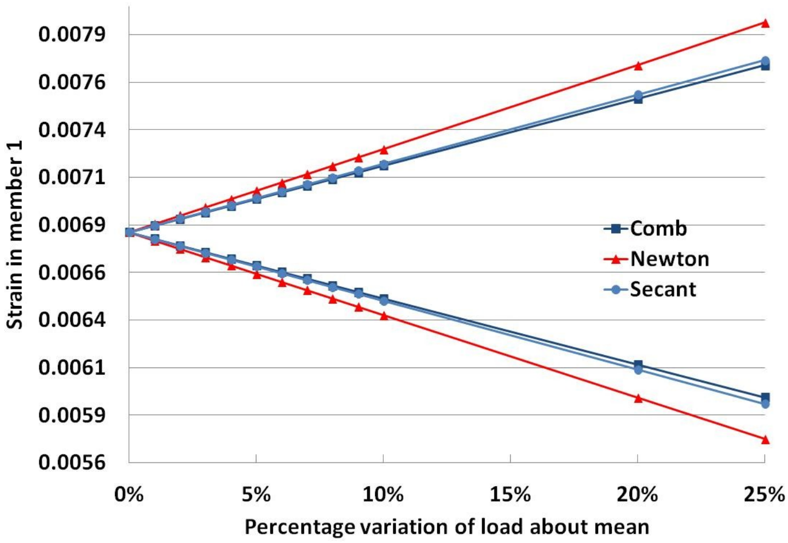

Figure 7. Five bar truss – variation of strain in member 1 with uncertainty of load

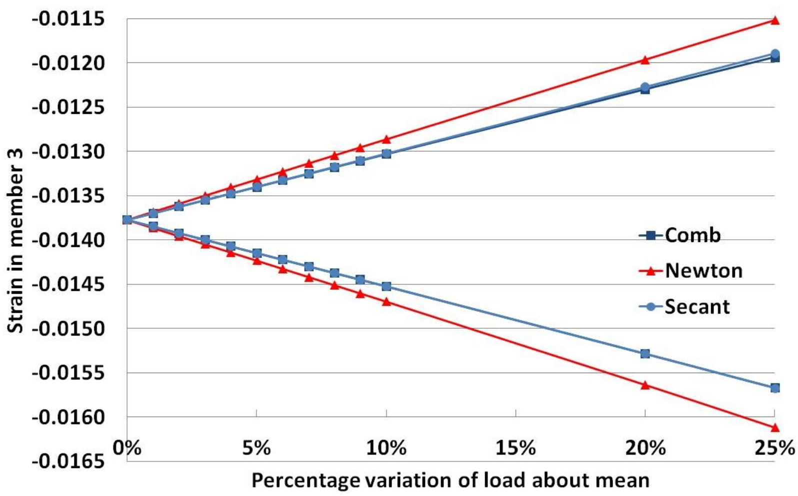

Figure 8. Five bar truss – variation of strain in member 3 with uncertainty of load

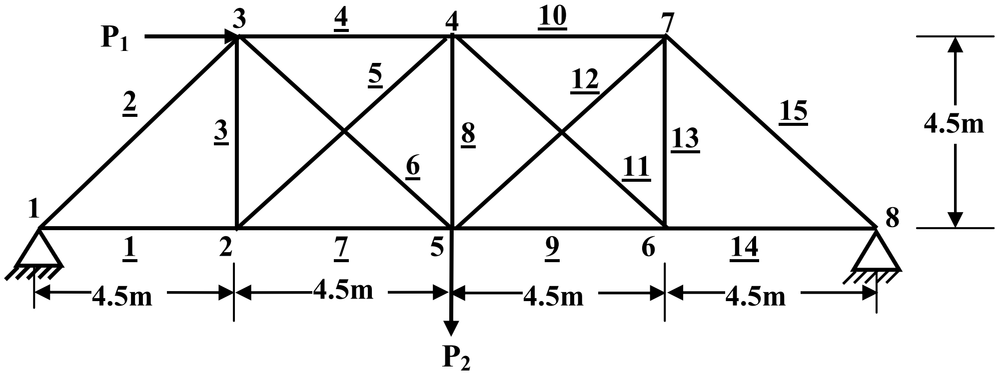

Figure 9 Fifteen-bar truss

The fifteen bar truss shown in Figure 9 is subjected to a horizontal point load P_1 = 150 kN acting to the right and vertical point load of P_2 = 250 kN acting downwards applied at the joints 3 and 5 respectively. Cross section areas of elements 1, 2, 3, 13, 14 and 15 are 10.0×10^{-5} m^2 while for the rest of the elements is the cross sectional area is 6.0×10^{-5} m^2. The deterministic value of Young's modulus of each element is $E_i = 2 \times 10^{11}$ N/m^2, $i = 1, 2, \ldots 15$. Results are computed using combinatorial approach, secant modulus approach and modified Newton-Raphson approach. Tables IV and V present the selected values of displacements and strains at load uncertainties of 1% and 10% respectively. It is observed from these tables that the displacements and strains computed using the modified Newton-Raphson approach and secant modulus approach are sharply enclosing the corresponding values computed using combinatorial solution at all levels of uncertainty. Figures 10 and 11 shows the plot of strain in members 2 and 8 computed for various levels of uncertainty from 0% to 10%. Figures 12 and 13 show the plots of horizontal displacement at node 3 and vertical displacement at node 5 respectively, computed for various levels of uncertainty from 0% to 10% .It is observed in all these figures that the solution computed using secant modulus approach as well as modified Newton-Raphson approach enclose the combinatorial solution at all levels of uncertainty.

Table IV Fifteen bar truss – Selected values of displacements and strains for 1% uncertainty in load

Method	U_3 (m)		$V_5 \times 10^1$ (m)		$\varepsilon_2 \times 10^3$		$\varepsilon_8 \times 10^2$	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	1.38087	1.39857	-3.52317	-3.48189	-6.29427	-6.17663	1.25047	1.26541
Secant	1.38079	1.39863	-3.52323	-3.48179	-6.29427	-6.17662	1.25026	1.26561
Error%	0.0058	0.0043	0.0017	0.0029	0.000	0.0002	0.0168	0.0002
Newton	1.37204	1.40475	-3.53113	-3.47009	-6.37471	-6.09618	1.24730	1.26716
Error%	0.6395	0.4419	0.2259	0.3389	1.2780	1.3025	0.2535	0.1383

Table V Fifteen bar truss – Selected values of displacements and strains for 10% uncertainty in load

Method	U_3 (m)		$V_5 \times 10^1$ (m)		$\varepsilon_2 \times 10^3$		$\varepsilon_8 \times 10^2$	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial	1.30352	1.48091	-3.71321	-3.29977	-6.82498	-5.64853	1.18457	1.33427
Secant	1.29957	1.48158	-3.71391	-3.29394	-6.82499	-5.64678	1.18055	1.33637
Error%	0.3030	0.0452	0.0189	0.1767	0.0001	0.0310	0.3394	0.1574
Newton	1.22645	1.55402	-3.80985	-3.19791	-7.63374	-4.84061	1.15888	1.35789
Error%	5.9125	4.9368	2.6026	3.0869	11.8500	14.3032	2.1687	1.7703

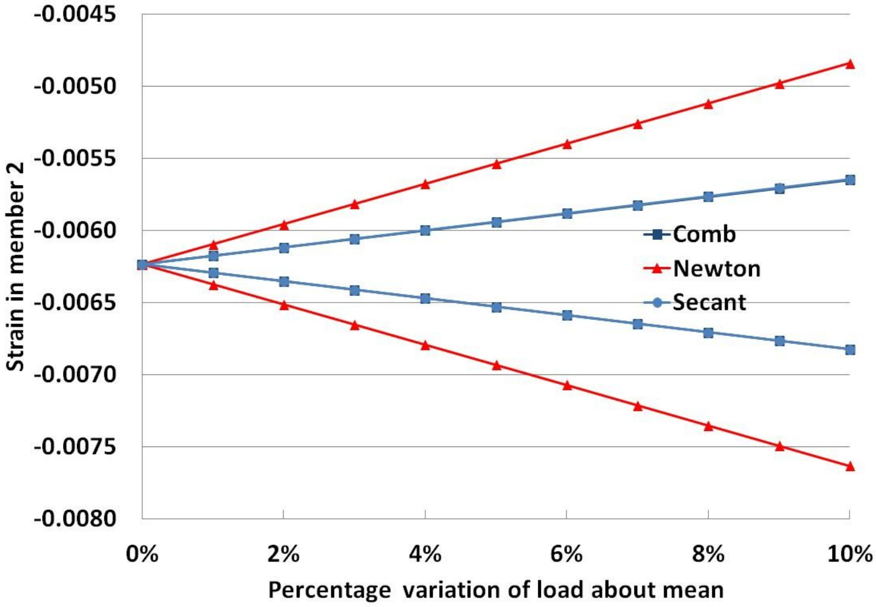

Figure 10. Fifteen bar truss – Variation of strain in member 2 with uncertainty of load

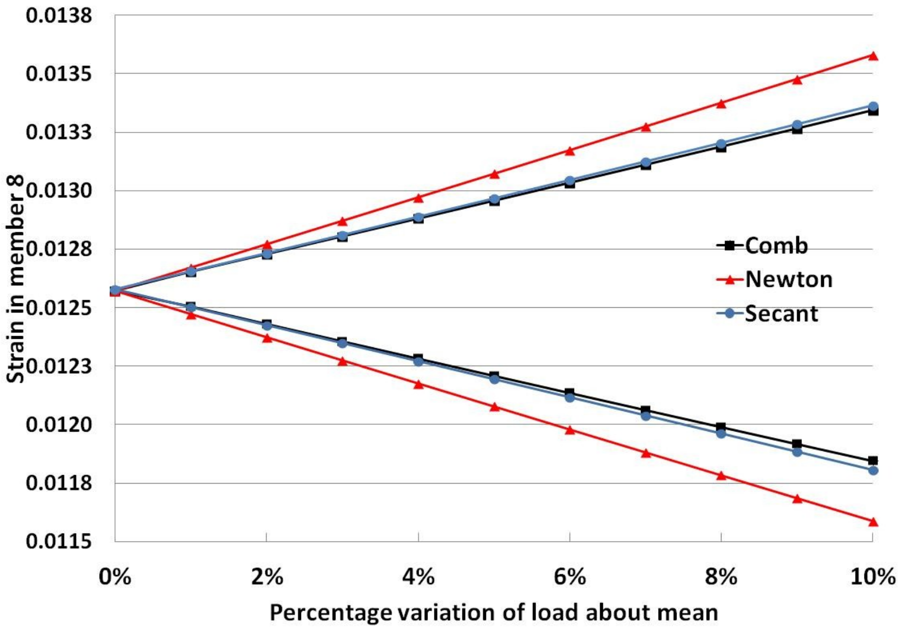

Figure 11. Fifteen bar truss – Variation of strain in member 8 with uncertainty of load

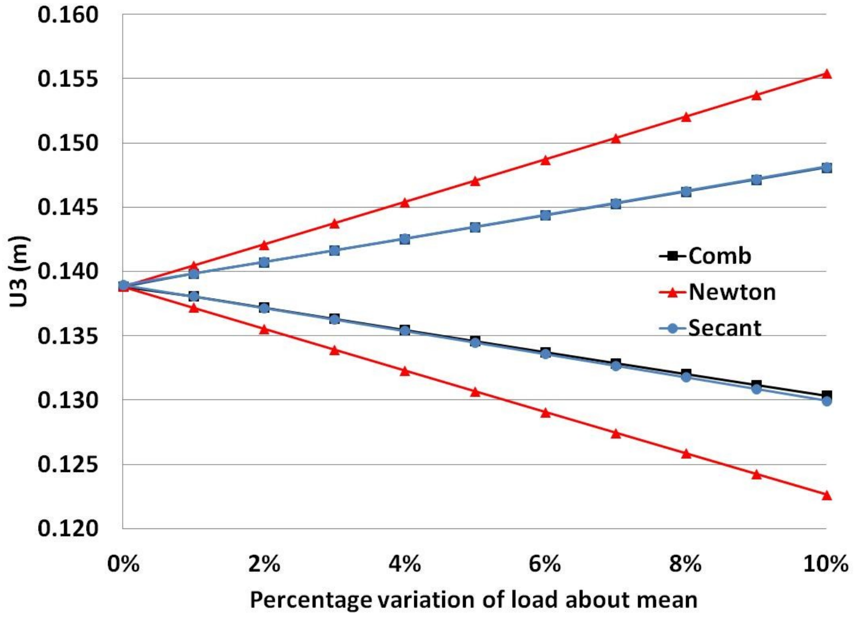

Figure 12. Fifteen bar truss – Variation of horizontal displacement at node 3 with uncertainty of load

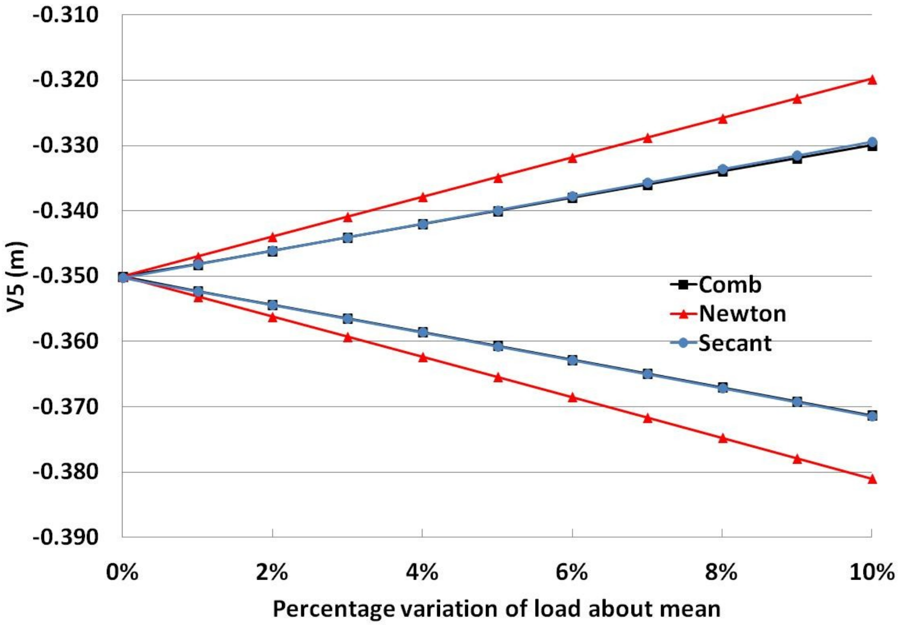

Figure 13. Fifteen bar truss – Variation of vertical displacement at node 5 with uncertainty of load

Conclusion

A Non-linear Interval Finite Element Method (NIFEM) for structural analysis is presented. Uncertainty in the applied load is represented as interval numbers and material nonlinearity is considered. The presented methods are an interval extension of the well known modified Newton-Raphson and the secant methods. Example problems illustrate the application of the methods to truss problems with large uncertainties. A new containment stopping criterion, which is intrinsic for intervals, has been introduced. The computational cost of the extension to interval numbers in both methods is comparable to the additional cost associated with introducing interval values into linear problems (Muhanna, *et al.*, 2007). Further study of non-linear interval finite elements methods for the refinement for different nonlinear material models is still needed to provide a more complete understanding of nonlinear interval finite element methods.

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Estimating load condition having caused structure failure and an optimal design taking account of the estimated result

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Abstract: In order to improve a design of structure, it is important to know the actual load condition of failed structure. We develop an estimation method of loading conditions based on images of failed structures and an FEM analysis model. Preparing a database that consists of deformation data of the structure corresponding to various load conditions, our system is able to estimate the load conditions that caused structure failure based on the processed images of failed structure samples. Adopting elasto-plastic model of the structure, the magnitude of the load having caused the failure is also estimated in addition to the position and orientation of the critical load. We adopt the EM algorithm to obtain the distribution of the critical load. An optimal design problem that takes account of the distribution of the estimated critical load condition is formulated as a minimization problem with a multi-objective function; the stiffness and the structural weight are also adopted as the evaluation items that make up the objective function. The particle swarm optimization (PSO) is adopted as the optimization algorithm. The approach is applied to crane-hook. The result of estimated critical load distribution and the optimal design based on the load distribution are demonstrated.

Keywords: load estimation; optimal design; database; finite element analysis; EM algorithm; crane-hook.

1. Introduction

Avoiding failure of structure system is one of the most important missions for design engineer of structures. In order to improve an existing structure so that it does not fail, it is important to know the load condition that causes structure failure. Generally, the load condition is identified by integrating the information obtained from the sensor devices; continuous monitoring is essential. However, almost all structures themselves have no information about the load conditions during their service life. In this case, several failure detection methods proposed in the past are not applicable (Quek et al., 2009; Lam and Ng, 2009). It is necessary to estimate the load condition by means of another approach. We develop a load estimation system; this system is applicable to the failed structures having permanent deformation. The system inputs the digital images of failed structures and outputs the estimated probability of the load conditions that caused the failure. The information from the sensor devices is not required. We deal with the failed crane-hooks as a concrete example.

Crane-hook is one of the most useful equipments for suspension work. Recently, excavators having a

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